Lecture

Music Processing

Meinard Müller

University of Erlangen-Nuremberg
International Audio Laboratories Erlangen
meinard.mueller@audiolabs-erlangen.de

Dynamic Time Warping (DTW)

Solution to Problem 1

$$\mathrm{DTW}(X,Y)=3,\quad \mathrm{DTW}(X,Z)=3,\quad \mathrm{DTW}(Y,Z)=3.$$

Solution to Problem 2

$$C=\begin{pmatrix}5&4&4&1\\3&2&2&3\\3&2&2&3\\6&5&5&0\\0&1&1&6\end{pmatrix}\qquad D=\begin{pmatrix}17&13&13&9\\12&9&9&8\\9&7&7&5\\6&5&6&2\\0&1&2&8\end{pmatrix}$$

An optimal warping path is given by

$$((1,1),(1,2),(1,3),(2,4),(3,4),(4,4),(5,4)).$$

There is no other optimal warping path.

Solution to Problem 3

Let $X=(x_1,x_2,\ldots,x_N)$ and $Y=(y_1,y_2,\ldots,y_M)$ be two arbitrary sequences over $\mathcal{F}$. Furthermore, let $p=(p_1,\ldots,p_L)$ with $p_\ell=(n_\ell,m_\ell)\in[1:N]\times[1:M]$, $\ell\in[1:L]$, be a warping path between X and Y with total cost

$$c_p(X,Y):=\sum_{\ell=1}^{L}c(x_{n_\ell},y_{m_\ell}).$$

We define a path $q=(q_1,\ldots,q_L)$ by $q_\ell:=(m_\ell,n_\ell)\in[1:M]\times[1:N]$. Obviously, q defines a warping path between Y and X. Furthermore, because of the symmetry of c, one has $c_q(Y,X)=c_p(X,Y)$. It follows that q is an optimal warping path between Y and X if and only if p is an optimal warping path between X and Y. Hence, $\mathrm{DTW}(X,Y)=\mathrm{DTW}(Y,X)$.

Let $\mathcal{F}=\{\alpha,\beta,\gamma\}$ and $c:\mathcal{F}\times\mathcal{F}\to\mathbb{R}_{\geq 0}$ defined by $c(x,y):=1-\delta_{xy}$. Furthermore, let $X=\alpha\beta\gamma$, $Y=\alpha\beta\beta\gamma$ and $Z=\alpha\gamma\gamma$. Then $\mathrm{DTW}(X,Y)=\mathrm{DTW}(Y,X)=0$, $\mathrm{DTW}(X,Z)=1$ and

$$\mathrm{DTW}(Y,Z)=2>1=\mathrm{DTW}(Y,X)+\mathrm{DTW}(X,Z).$$

Hence, the triangular inequality is violated.

Solution to Problem 4

Let L_k be the length of the warping path at level k between the sequences X_k and Y_k each having length 2^{n-k+1}, $1 \leq k \leq n$. Then,

$$L_k \leq 2 \cdot 2^{n-k+1} = 2^{n-k+2},$$

i. e., $L_1 \leq 2N$, $L_2 \leq 2(N/2) = N$, and so on. Then the following holds:

$$\begin{aligned}
A^{\text{MsDTW}}(N) &= A^{\text{MsDTW}}\left(\frac{N}{2}\right) + f_1^2 \cdot L_2 \\
&\leq A^{\text{MsDTW}}\left(\frac{N}{2}\right) + 4 \cdot N \\
&= A^{\text{MsDTW}}\left(\frac{N}{2^2}\right) + f_2^2 \cdot L_3 + 4 \cdot N \\
&\leq A^{\text{MsDTW}}\left(\frac{N}{2^2}\right) + 4 \cdot \left(\frac{N}{2}\right) + 4 \cdot N \\
&\leq \ldots \\
&\leq A^{\text{MsDTW}}\left(\frac{N}{2^{n-1}}\right) + 4 \cdot \left(\frac{N}{2^{n-2}}\right) \ldots + 4 \cdot \left(\frac{N}{2^2}\right) + 4 \cdot \left(\frac{N}{2}\right) + 4 \cdot N \\
&\leq A^{\text{DTW}}(2) + 4 \cdot \left(4 + \ldots + 2^{n-2} + 2^{n-1} + 2^n\right) \\
&\leq 4 \cdot \sum_{k=0}^{n} 2^k \\
&\leq 4 \cdot 2^{n+1} \\
&= 8N
\end{aligned}$$

The last inequality can be shown easily via induction.