Lecture

Music Processing

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Fourier Transform

Exercise 1

Let $f : [0,1] \to \mathbb{R}$, $f \in L^2([0,1])$. Then, f can be represented in form of the following two variants of the Fourier series:

$$f(t) = d_0 + \sum_{k\in\mathbb{N}} d_k \sqrt{2}\cos(2\pi(kt - \varphi_k)) \tag{1}$$

$$f(t) = a_0 + \sum_{k\in\mathbb{N}} a_k \sqrt{2}\cos(2\pi kt) + \sum_{k\in\mathbb{N}} b_k \sqrt{2}\sin(2\pi kt) \tag{2}$$

Show that $d_0 = a_0$, $d_k = \sqrt{a_k^2 + b_k^2}$, and $\varphi_k = \frac{1}{2\pi}\arccos\left(\frac{a_k}{d_k}\right)$ for $k \in \mathbb{N}$.

Solution

We use the addition theorem $\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$ with $\alpha = 2\pi kt$, $\beta = -2\pi\varphi_k$. Then

$$\begin{aligned}\cos(2\pi(kt-\varphi_k)) &= \cos(2\pi kt)\cos(-2\pi\varphi_k) - \sin(2\pi kt)\sin(-2\pi\varphi_k)\\ &= \cos(2\pi\varphi_k)\cos(2\pi kt) + \sin(2\pi\varphi_k)\sin(2\pi kt)\end{aligned}$$

Comparing coefficients in (1) and (2) one obtains

$$\left.\begin{array}{lcl} a_k & = & d_k\cos(2\pi\varphi_k)\\ b_k & = & d_k\sin(2\pi\varphi_k)\end{array}\right\} \Rightarrow (a_k^2 + b_k^2) = d_k^2,$$

and $\cos(2\pi\varphi_k) = \frac{a_k}{d_k}$ implies $\varphi_k = \frac{1}{2\pi}\arccos\left(\frac{a_k}{d_k}\right)$.

Exercise 2

Let $f : [0,1] \to \mathbb{R}$, $f \in L^2([0,1])$, be as in the previous exercise. Then, using a complex formulation of the Fourier series, f can be represented as

$$f(t) = \sum_{k\in\mathbb{Z}} c_k e^{2\pi ikt}. \tag{3}$$

Let $c_k = |c_k|e^{2\pi i\gamma_k}$ be the polar coordinate representation of the complex number c_k. Show that

$$\begin{aligned} &d_0 = c_0, \quad d_k = \sqrt{2}|c_k|, \quad \varphi_k = \gamma_k, \\ &a_k = \sqrt{2}\operatorname{Re}(c_k), \quad b_k = -\sqrt{2}\operatorname{Im}(c_k). \end{aligned}$$

for $k \in \mathbb{N}$.

Solution

Since f is a real-valued function, one obtains

$$\sum_{k\in\mathbb{Z}} c_k e^{2\pi ikt} = f(t) = \overline{f(t)} = \overline{\sum_{k\in\mathbb{Z}} c_k e^{2\pi ikt}} = \sum_{k\in\mathbb{Z}} \overline{c_k} e^{-2\pi ikt}$$

and therefore $c_{-k} = \overline{c_k}$. From this follows

$$\begin{aligned} f(t) &= \sum_{k\in\mathbb{Z}} c_k e^{2\pi ikt} \\ &= c_0 + \sum_{k\in\mathbb{N}} \left(c_k e^{2\pi ikt} + c_{-k} e^{-2\pi ikt}\right) \\ &= c_0 + \sum_{k\in\mathbb{N}} \left(c_k e^{2\pi ikt} + \overline{c_k}\,\overline{e^{2\pi ikt}}\right) \\ &= c_0 + \sum_{k\in\mathbb{N}} 2\operatorname{Re}(c_k e^{2\pi ikt}) \\ &= c_0 + \sum_{k\in\mathbb{N}} 2\operatorname{Re}\Big((\operatorname{Re}(c_k) + i\operatorname{Im}(c_k)) \cdot (\cos(2\pi kt) + i\sin(2\pi kt))\Big) \\ &= c_0 + \sum_{k\in\mathbb{N}} 2\operatorname{Re}(c_k)\cos(2\pi kt) - 2\operatorname{Im}(c_k)\sin(2\pi kt). \end{aligned} \tag{4}$$

Hence, by comparing coefficients of (4) with (2), one obtains

$$\begin{aligned} a_k &= \sqrt{2}\operatorname{Re}(c_k), \quad b_k = -\sqrt{2}\operatorname{Im}(c_k) \\ d_k &= \sqrt{a_k^2 + b_k^2} = \sqrt{2}|c_k|. \\ d_0 &= c_0. \end{aligned}$$

Moreover, by substituting a_k and d_k, one obtains

$$\begin{aligned} \varphi_k &= \frac{1}{2\pi}\arccos\left(\frac{a_k}{d_k}\right) \\ &= \frac{1}{2\pi}\arccos\left(\frac{\operatorname{Re}(c_k)}{|c_k|}\right) \\ &= \frac{1}{2\pi}\arccos\left(\frac{|c_k|\cos(2\pi\gamma_k)}{|c_k|}\right) = \gamma_k. \end{aligned}$$