④ Complex numbers and exponential function

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Complex numbers $\mathbb{C} = \mathbb{R}^2$, $i = \sqrt{-1}$

p.45
Fig. 2.4

$c = Re(c) + i \cdot Im(c) = a + ib$

$\bar{c} := Re(c) - i\, Im(c)$

$Re(c) = (c + \bar{c})/2$, $Im(c - \bar{c})/2$

$|c| = \left(R(c)^2 + Im(c)^2 \right)^{\frac{1}{2}}$

Exponential function

p.72
Fig. 2.17

$\exp(i\gamma) = \cos(\gamma) + i \sin(\gamma)$, $\gamma \in \mathbb{R}$

$\exp(i(\gamma + 2\pi)) = \exp(i\gamma)$

$|\exp(i\gamma)| = 1$

$\overline{\exp(i\gamma)} = \exp(-i\gamma)$

$\exp(i(\gamma_1 + \gamma_2)) = \exp(i\gamma_1)\, \exp(i\gamma_2)$

Trigonometric identities follow!

$\exp(i(\gamma_1 + \gamma_2)) = \cos(\gamma_1 + \gamma_2) + i \sin(\gamma_1 + \gamma_2)$

$\shortparallel$

$\exp(i\gamma_1)\exp(i\gamma_2) = (\cos(\gamma_1) + i\sin(\gamma_1))\,(\cos(\gamma_2) + i\sin(\gamma_2))$

$= (\cos(\gamma_1)\cos(\gamma_2) - \sin(\gamma_1)\sin(\gamma_2)) + i\,(\cos(\gamma_1)\sin(\gamma_2) + \cos(\gamma_2)\sin(\gamma_1))$

p.72
Fig. 2.17

Polar coordinates of complex numbers:

$c = a + ib = Re(c) + i\, Im(c)$

$c = |c| \cdot \exp(i\gamma)$

$\gamma = \operatorname{atan2}(b, a)$ "=" $\arctan\left(\frac{b}{a}\right) = \arctan\left(\frac{Im(c)}{Re(c)}\right)$

$Re(c) = |c| \cos(\gamma)$

$Im(c) = |c| \cdot \sin(\gamma)$

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⑤ Fourier transform as "optimal solution"

Fourier representation: $f(t) = \int_{\omega \in \mathbb{R}_{\geq 0}} d_\omega \sqrt{2}\cos(2\pi(\omega t - \varphi_\omega))\, d\omega$

Fourier transform: $c_\omega := \hat{f}(\omega) := \int_{t\in\mathbb{R}} f(t)\exp(-2\pi i\omega t)\,dt = |c_\omega| \exp(i\gamma_\omega)$

Claim: $d_\omega := \max_{\varphi\in[0,1]} \left(\int_t f(t)\sqrt{2}\cos(2\pi(\omega t - \varphi))\,dt \right) = \sqrt{2}|c_\omega|$

$\varphi_\omega := \operatorname{argmax}_{\varphi\in[0,1]} \left(\int_t f(t)\sqrt{2}\cos(2\pi\omega t - \varphi)\right) dt = -\frac{1}{2\pi}\gamma_\omega$

Proof: $g: \mathbb{R} \to \mathbb{R}$

$$g(\varphi) := \int_{t\in\mathbb{R}} f(t)\sqrt{2}\cos(2\pi\omega t - 2\pi\varphi))\,dt$$
$$= \sqrt{2}\int_{t\in\mathbb{R}} f(t)\left(\cos(2\pi\omega t)\cos(-2\pi\varphi) - \sin(2\pi\omega t)\sin(-2\pi\varphi)\right)dt$$
$$= \sqrt{2}\cos(2\pi\varphi)\underbrace{\int_{t\in\mathbb{R}} f(t)\cos(2\pi\omega t)}_{Re(c_\omega)} + \sqrt{2}\sin(2\pi\varphi)\underbrace{\int_{t\in\mathbb{R}} f(t)\sin(2\pi\omega t)\,dt}_{-Im(c_\omega)}$$

$$\frac{\partial g}{\partial\varphi}\Big|_\varphi = -\sqrt{2}\,2\pi\sin(2\pi\varphi)Re(c_\omega) - \sqrt{2}\,2\pi\cos(2\pi\varphi)Im(c_\omega)$$

$$\frac{\partial g}{\partial\varphi}\Big|_\varphi = 0 \iff -\sin(2\pi\varphi)Re(c_\omega) = \cos(2\pi\varphi)Im(c_\omega)$$
$$\iff \frac{\sin(-2\pi\varphi)}{\cos(-2\pi\varphi)} = \frac{Im(c_\omega)}{Re(c_\omega)}$$
$$\iff \tan(-2\pi\varphi) = \frac{Im(c_\omega)}{Re(c_\omega)}$$
$$\iff \varphi = -\frac{1}{2\pi}\arctan\left(\frac{Im(c_\omega)}{Re(c_\omega)}\right)$$
$$\implies \varphi = -\frac{1}{2\pi}\gamma_\omega$$

Hence: Optimal $\varphi_\omega = -\frac{1}{2\pi}\gamma_\omega$

$$g(\varphi_\omega) = \sqrt{2}\cos(2\pi(-\tfrac{1}{2\pi}\gamma_\omega)Re(c_\omega) + \sqrt{2}\sin(2\pi(-\tfrac{1}{2\pi}\gamma_\omega))(-Im(c_\omega))$$
$$= \sqrt{2}\cos(\gamma_\omega)Re(c_\omega) + \sqrt{2}\sin(\gamma_\omega)Im(c_\omega)$$
$$= \sqrt{2}\frac{Re(c_\omega)}{|c_\omega|}\cdot Re(c_\omega) + \sqrt{2}\frac{Im(\gamma_\omega)}{|c_\omega|}Im(c_\omega)$$
$$= \sqrt{2}\left(Re(c_\omega)^2 + Im(c_\omega)^2\right)\cdot\frac{1}{|c_\omega|}$$
$$= \sqrt{2}|c_\omega|^2\frac{1}{|c_\omega|} = \sqrt{2}|c_\omega|$$

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Complex and real versions of Fourier representation

Let $f: \mathbb{R} \to \mathbb{R}$ be a real-valued signal, then

1) $f(t) = \int_{\omega \geq 0} d_\omega \sqrt{2} \cos(2\pi(\omega t - \varphi_\omega))\, d\omega$

2) $f(t) = \int_{\omega \in \mathbb{R}} c_\omega \exp(2\pi i \omega t)\, d\omega$

with $d_\omega = \sqrt{2}\, |c_\omega|$ and $\varphi_\omega = -\frac{1}{2\pi} \gamma_\omega$

Lemma: $c_{-\omega} = \overline{c_\omega}$ when f is real-valued

Proof:
$$c_{-\omega} = \int_{t \in \mathbb{R}} f(t) \exp(2\pi i \omega t)\, dt$$
$$= \int_{t \in \mathbb{R}} \overline{f}(t)\, \overline{\exp(-2\pi i \omega t)}\, dt \quad (\text{since } f = \overline{f})$$
$$= \overline{\int_{t \in \mathbb{R}} f(t) \exp(-2\pi i \omega t)\, dt} = \overline{c_\omega}$$

$$\int_{\omega \in \mathbb{R}} c_\omega \exp(2\pi i \omega t)\, d\omega$$
$$= \int_{\omega \geq 0} c_\omega \exp(2\pi i \omega t)\, d\omega + \int_{\omega \geq 0} c_{-\omega} \exp(2\pi i (-\omega) t)\, d\omega$$

$c_{-\omega} = \overline{c_\omega}$
$$= \int_{\omega \geq 0} c_\omega \exp(2\pi i \omega t)\, d\omega + \int_{\omega \geq 0} \overline{c_\omega \exp(2\pi i \omega t)}\, d\omega$$
$$= \int_{\omega \geq 0} 2\,\mathrm{Re}(c_\omega \exp(2\pi i \omega t))\, d\omega$$
$$= \int_{\omega \geq 0} 2\,\mathrm{Re}(c_\omega) \cos(2\pi \omega t) - 2\,\mathrm{Im}(c_\omega) \sin(2\pi \omega t)\, d\omega$$
$$= \int_{\omega \geq 0} 2\,|c_\omega| \cos(\gamma_\omega) \cos(2\pi \omega t) - 2\,|c_\omega| \sin(\gamma_\omega) \sin(2\pi \omega t)\, d\omega$$
$$= \int_{\omega \geq 0} \sqrt{2} d_\omega \cos(-2\pi \varphi_\omega) \cos(2\pi \omega t) - \sqrt{2} d_\omega \sin(-2\pi \varphi_\omega) \sin(2\pi \omega t)\, d\omega$$
$$= \int_{\omega \geq 0} d_\omega \sqrt{2} \left(\cos(2\pi \omega t) \cos(2\pi \varphi_\omega) + \sin(2\pi \omega t) \sin(2\pi \varphi_\omega) \right) d\omega$$
$$= \int_{\omega \geq 0} d_\omega \sqrt{2} \cos(2\pi(\omega t - \varphi_\omega))\, d\omega$$