Book review

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Gareth E. Roberts. *From music to mathematics: Exploring the connections.* Baltimore, MD: Johns Hopkins University Press, 2016. 320 pp. ISBN: 9781421419183.

There has been a long-standing discussion about the connections between music and mathematics. Music involves patterns and structures that may be described using mathematical language. Some compositions are even constructed around certain mathematical ideas, and composers have used various kinds of symmetries and transformations to shape their works. Many concepts from music theory can be expressed using some basic mathematical formalisms. And of course, in the study of musical sounds, the relation between music and mathematics becomes quite obvious. In his book *From Music to Mathematics*, Gareth Roberts explores some of the connections between these two disciplines. Organized into eight chapters, the book covers a range of topics starting with well-known relationships such as the Pythagorean theory of musical scales and simple ratios, harmonic consonance and overtones series, as well as musical symmetries and group theory. More curious connections exist in scenarios such as change ringing, twelve-tone music, or mathematically inspired modern music.

The book is written with care, and in it Roberts reveals his passion for both music and mathematics. Each chapter starts with a certain musical aspect which leads to a mathematical problem. This problem is then formalized and treated in more detail. Reading through the book is like listening to a pleasant medley, in which one encounters some favorite tunes as well as new, surprising perspectives. Beyond simply pointing out interesting connections between music and mathematics, Roberts notes that one main goal of this book is to use music in order to illuminate important mathematical concepts. By doing so, the author tries to overcome the first hurdle many students are confronted with when studying abstract mathematics. Although this book is not meant to replace a proper textbook on algebra, number theory, combinatorics, trigonometry, or differential calculus, it does not refrain from using proper mathematical notation, all the while giving students a glimpse into the mathematical realm and its beauty. Rather than giving elaborate introductions in the different mathematical fields, Gareth Roberts covers different topics in an anecdotal and elementary form, which makes the book accessible for a wide readership including undergraduate and even advanced high school students. The following paragraphs will address the individual chapters of the book.

Music is typically organized into temporal units or pulses, referred to as beats. Repeating sequences of stressed and unstressed beats and sub-beats, in turn, form higher temporal patterns, which are related to what is called the *rhythm* of music. In Chapter 1, the book relates the musical notions of beat and rhythm to the fundamental mathematical concept of counting. First it discusses the role of note durations, which are specified in terms of rational numbers multiplied by the underlying beat duration.

Looking at the basic note types (whole, half, quarter, eighth, etc.) leads the writer to the mathematical concept of *geometric series*, which is then discussed in greater detail. Furthermore, the notion of a time signature (or meter), which is specified by two numbers that indicate the dominating note type and the number of beats per measure, is introduced. Superimposing two different rhythms creates *polyrhythms*, which are often encountered in African or Indian music. Determining relative note positions in polyrhythmic music recalls the mathematical concepts of *least common multiple* and *greatest common divisor*. Finally, counting the number of specific beat patterns that occur in Indian classical music results in the famous *Fibonacci numbers*. The chapter is nicely complemented by the discussion of numerous musical examples. For readers who want to dive deeper into the topic of this chapter the book *Rhythm and Transforms* by William Sethares (2007) is a good complementary source.

In Chapter 2, some fundamental aspects from harmony and counterpoint theory are summarized. Starting with some basics on Western sheet music notation, the author then reviews the fundamental concepts of scales, keys, intervals, and chords. Based on the circle of fifths, tonal proximity and the relation between major and minor, various musical keys are discussed. The chapter closes with a synopsis of the evolution of polyphony in the Western music tradition. Most of the material covered in this chapter can be found in standard introductory textbooks on music notation and harmony theory.

Music can be represented in many different ways and formats. For example, a composer may write down a composition in the form of a musical score, where musical symbols are used to visually encode notes and show how these notes are to be played by a musician. When musicians start delving into the music, the playing instructions recede into the background. The musical meter turns into a rhythmic flow; the different note objects melt into harmonic sounds and smooth melody lines; and the instruments communicate with each other (Müller, 2015). From a physical point of view, performing music results in *sounds* or *acoustic waves*, which are transmitted through the air as pressure oscillations. While the first two chapters cover musical aspects that can be described on the symbolic level, Gareth Roberts discusses in Chapter 3 some basic material on musical sounds and their properties. In particular, he explains the attributes of loudness, pitch, duration, and timbre from both a physical and perceptual point of view. Human perception of sound intensity is logarithmic in nature, which motivates the logarithmic *decibel* scale. The standard unit to measure frequency, or the number of cycles a wave makes in a second, is a *Hertz*. A sinusoid is the simplest type of periodic waveform, which leads into a branch of mathematics known as *trigonometry*. Using fundamental trigonometric identities, phenomena such as *beating* are explained. Furthermore, Roberts shows how sound production via vibrating objects leads to the realm of partial differential equations. The study of such equations yields a natural approach to sinusoids and also explains the phenomenon of *overtones* – sound components that are integer multiples of the fundamental frequency. The chapter closes with an instructive experiment that investigates the relation between length and pitch of a one-stringed instrument.

The experiment at the end of Chapter 3 and the overtone series naturally lead to another important and well-studied topic, tuning and temperament, which is covered in Chapter 4. The oldest known tuning system was introduced by the Greek philosopher and mathematician Pythagoras. Based only on the frequency ratios 2:1 (octave) and 3:2 (perfect fifth), all other intervals in the Pythagorean tuning are derived from these ratios by suitably adding and subtracting fifths and octaves. Defining a twelve-tone scale using only perfect fifths results in a small inconsistency known as *Pythagorean comma*. This fact can be nicely explained by considering the spiral of fifths – a never-ending extension of the circle of fifths. Besides the Pythagorean tuning, there are many more tuning systems that may be used for defining intervals in terms of frequency ratios. Another such system is known as *just intonation*, where intervals are defined by ratios that are well aligned with the overtone series. Similar to the Pythagorean tuning, just intonation also results in an inconsistency expressed by the *syntonic comma*. The twelve-tone *equal-tempered scale*, in which an octave is subdivided into twelve scale steps, can be considered as a compromise to address the deficiencies in the Pythagorean tuning and in just intonation. The fundamental frequencies of these twelve-scale steps are equally spaced on a logarithmic frequency axis. Even though equal temperament discards the harmonically pleasing ratios of integers, it results in a practical tuning system with consistent half steps and a closed circle of fifths. In this detailed chapter, the author also takes the opportunity to discuss rational and irrational numbers. Considering the special scenario of Strähle's Guitar, he introduces the mathematical concepts of linear fractional transformations and discusses the fascinating approximation properties of continued fractions. This nicely written chapter, which sheds new light on some well-known topics, is one of the highlights of the book.

In Chapter 5, going back from sounds to the symbolic music domain, Roberts discusses universal structures based on symmetry – principles that can be found in various musical forms and styles. In particular, he emphasizes the transformations known as translation, retrograde, inversion, and retrograde-inversion and illustrates these principles with a number of concrete music examples, including Bach's *Musical Offering*. Analyzing abstract properties of symmetry operations leads to the notion of a *group*, which is a fundamental concept in a mathematical branch known as *algebra*. After giving a formal definition, he discusses various geometrically motivated groups, including the so-called *dihedral group* and the one defined by the musical symmetries based on retrograde and inversion. As another universal structure, the *golden ratio* and its relation to the Fibonacci numbers are discussed. It is shown that these principles may have some relevance for Bartók's *Music for Strings, Percussion, and Celesta*.

Chapter 6 introduces a rather unusual, yet intriguing connection between a musical activity and algebra. The art of *change ringing* refers to ringing a set of particular bells in a systematic manner to produce variations in their sounding order. A *change* is a specific arrangement of the bells so that each bell is rung exactly once. This leads to the mathematical notion of a *permutation*, which relates to the act of rearranging the distinct bells into some sequence or order. As detailed in this chapter, one of the primary goals in change ringing is to perform a number of changes that conform to a given set of rules. In particular, starting with an initial change, each possible change (or permutation) should be covered exactly once while imposing restrictions between successive changes before returning to the initial change. These rules cause challenging combinatorial problems, which are discussed explicitly for some specific examples. Motivated by this musical scenario, Roberts then resumes the topic of abstract group theory. The set of all permutations of a fixed size (corresponding to the number of distinct bells) forms one of the most important mathematical groups known as the *symmetric group*. Its importance is founded in the famous theorem by Cayley stating that *every* finite group is isomorphic to a subgroup of some finite symmetric group (an explanation that Roberts does not mention in this book). As a special case, it is shown how the dihedral group introduced in the previous chapter appears as a subgroup of the symmetric group of order four. In regards to the combinatorial problems occurring in change ringing, it is important at this point to mention another mathematically insightful approach not covered in this book. The permutations may be interpreted as the vertices of a graph, where two vertices are connected by an edge if there is a permitted transition (according to change ringing rules) that transforms one change into the other. Then a full transit through all possible changes corresponds to a so-called *Hamiltonian cycle*, a path that starts and ends at the same vertex while visiting each vertex in the graph exactly once (except the first and last one).

Chapter 7 is dedicated to a specific method of composition known as the *twelve-tone technique*. This technique, first proposed by Arnold Schoenberg, has had a great influence on many musical styles in the 20th and 21st centuries. Roberts introduces the main idea of the twelve-tone technique, in which a *tone row* consisting of a specific arrangement of the twelve chromatic pitch classes serves as the basic building block for a composition. Continuing the discussion from Chapter 5, he presents the *tone row matrix* as a concept for generating different transformations of a tone row. Then, the application of the twelve-tone concept is outlined by analyzing examples from Schoenberg's *Suite für Klavier* Op. 25. Finally, the chapter discusses invariance properties of tone rows under certain transformations. By carefully explaining certain symmetries in a step-by-step fashion, Roberts shows how to calculate the total number of distinct rows.

In the final Chapter 8, Roberts presents specific examples from modern music, which are constructed by explicitly using mathematical concepts. First, the music of the British composer Sir Peter Maxwell Davies is considered. In many of his works, so-called *magic squares* serve as construction principle. These are symmetrical matrices consisting of consecutive natural numbers, in which the sum of the elements in each row, each column, and in the two diagonals is the same. The numbers are used to determine pitches and note durations. By following contiguous paths in the magic squares, Davies generates rhythms and melodic lines used as building blocks in his compositions. In the next section, a work named *Clapping Music* by the composer Steve Reich is considered. This purely percussive piece for two musicians clapping their hands is generated from a single rhythmic pattern. Through a technique known as *phase shifting*, the pattern is repeated using different offsets to be applied by the two clapping musicians. Roberts highlights the composer's choice of the specific pattern by discussing its uniqueness. As a final example, two pieces by the Greek architect and composer Iannis Xenakis are presented. The first piece called *Metastasis* uses glissando elements to create a special surface. The other piece called *Pithoprakta* employs random processes – a principle from the field of *stochastics* – to generate clouds of sounds. The chapter closes with an instruction for students to compose their own music using mathematical concepts.

As discussed in the previous paragraphs, the book offers a selection of different topics in which music and mathematics intersect. Using music as a springboard, Roberts touches on various mathematical disciplines, such as algebra, number theory, combinatorics, trigonometry, and differential calculus. He does not hold back from using a mathematically clean formalism. At the same time, he confines himself to introducing only a small number of mathematical concepts, which serve as appetizers, giving a glimpse into the various mathematical disciplines. Furthermore, by discussing numerous explicit musical examples, Roberts manages to always connect back to the music domain. The selection of the book's topics is based on the author's teaching experience and his desire to provide a sufficiently comprehensive and practical source for instructors and students. Many of the chapters can be used modularly, which allows instructors in various disciplines to easily include material from the book within their courses.

A drawback of the modular structure is that a common thread throughout the book is not easy to find. Also, the fundamental question, to which extent music and mathematics are actually connected, remains unanswered. Surely, basic mathematics may help to better understand music theory – and some music may have been inspired by mathematics. One may argue that music and mathematics are intellectually related. However, does mathematics really help to better understand or appreciate music? How deep is the relation between music and mathematics? Is a composer really aware of mathematical structures when composing music? Is the relation between music and mathematics special, going beyond the connections between physics and mathematics, architecture and mathematics, or visual arts and mathematics? A deeper discussion of such questions would have further rounded off the book.

Of course, selecting topics for a textbook is always a matter of taste, reflecting an author's teaching experience and preferences. Still, in view of recent developments in the music industry and the way music is produced and consumed, a number of important topics that would have nicely fit into the scope of the book are missing. For example, the book does not cover important topics such as digitization and digital representations of music – issues that are closely connected to both music and mathematics. Also, electronic tools used to synthesize and process music are based on mathematical tools and transformations that could have been discussed in this book. Maybe the most important and well-known tool is the *Fourier transform*, which converts a musical signal that depends on time into a representation that depends on frequency. This topic could have been perfectly added after Chapter 3 on *The Science of Sound*, in which the connection between a musical sound and a sine function is discussed. These are precisely the building blocks used in *Fourier analysis*, where a musical signal is compared with sinusoids of various frequencies. This results in a decomposition that unfolds the frequency spectrum of the signal – similar to a prism that can be used to break light up into its constituent spectral colors.

Even though references to relevant literature are given for each chapter individually, a broader discussion of related work and links for further reading would have been desirable. There are numerous books that establish connections between music and mathematics. For example, many topics addressed in this book (e.g., music notation, scales, tuning, intonation, consonance, sound generation, acoustics, vibrating systems, wave equations, and computational methods for composition) are also covered by the more comprehensive two-volume work, *Musimathics: The Mathematical Foundations of Music* by Gareth Loy (2011a, 2011b). Similar to the book by Gareth Roberts, *Mathematics and Music* by David Wright (2009) covers rational numbers and musical intervals, scales and tuning, chromatic scales, and modular arithmetic. In his book *The Geometry of Musical Rhythm*, Godfried Toussaint (2013) provides a systematic and accessible computational geometric analysis of musical rhythms. *Music: A Mathematical Offering* by Dave Benson (2006) covers topics such as sound and Fourier analysis, consonance and dissonance, scales and temperaments, music synthesis, and symmetries in music. Topics such as tuning and temperament, Helmholtz' theory on consonance, or geometry of music can also be found in the book *Music and Mathematics: From Pythagoras to Fractals* edited by Fauvel, Flood, and Wilson (2006). The now-classical text *Gödel, Escher, Bach: An Eternal Golden Braid* by Douglas Hofstadter (1999) explains the connections of form, geometry, logic, recursion, formal systems, and artificial intelligence. Where Gareth Roberts' book stands among existing books remains unclear. Furthermore, *From Music to Mathematics* would have benefited by pointing out relations to more advanced literature and other disciplines such as computational mathematics, audio processing, or music information retrieval. For example, in his book *The Topos of Music*, Guerino Mazzola (2002) establishes connections between music, cognition, composition, and deep mathematical concepts from algebra and category theory. Interesting relations between music, mathematics, and computer science have become relevant in fields such as Music Information Retrieval (MIR), which systematically deals with a wide range of computer-based music analysis, processing, and retrieval topics. Certain chapters of Roberts' book offer suitable foundations for MIR textbooks such as *Fundamentals of Music Processing* by Meinard Müller (2015).

Overall, *From Music to Mathematics* is a pleasing and well-written book that is accessible for everyone who wants to explore the connections between music and mathematics. Gareth Roberts does a great job of making numerous suggestions on how music can be used to illuminate mathematical concepts. The selection of music examples covering a wide range of musical styles is appropriate for illustrating the relation between the two fields. Containing homework exercises as well as links to relevant literature at the end of each chapter, the book provides excellent material that can be easily included in more advanced courses in various disciplines. In this sense, the book makes some substantial pedagogical contributions. And last but not least, *From Music to Mathematics* is very enjoyable to read – not only for students, but for anyone who loves music and mathematics.

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Meinard Müller and Christof Weiß
International Audio Laboratories Erlangen, Germany