Fundamentals of Music Processing

Chapter 2: Fourier Analysis of Signals

Meinard Müller

International Audio Laboratories Erlangen

www.music-processing.de

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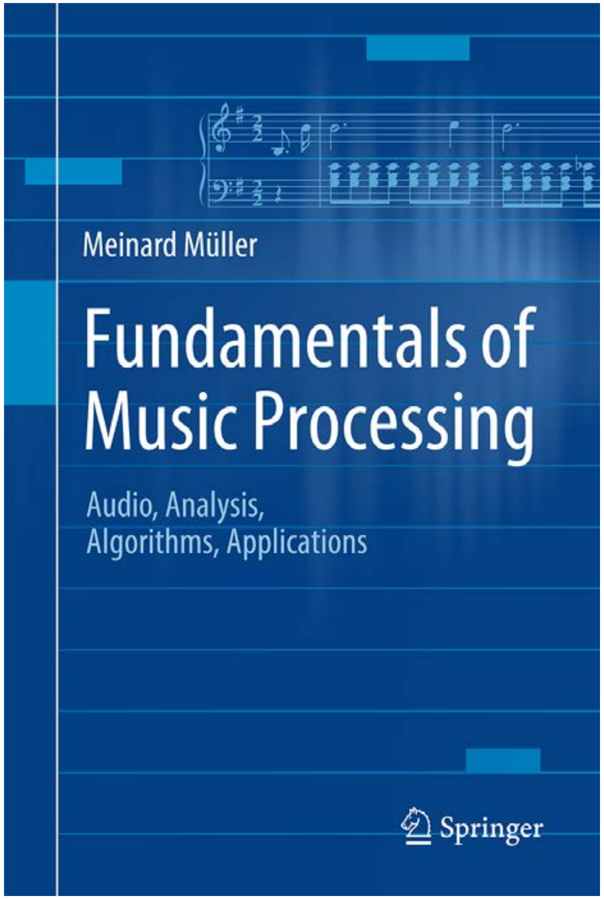
Book: Fundamentals of Music Processing



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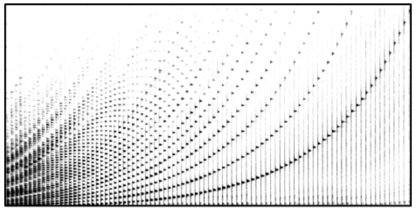
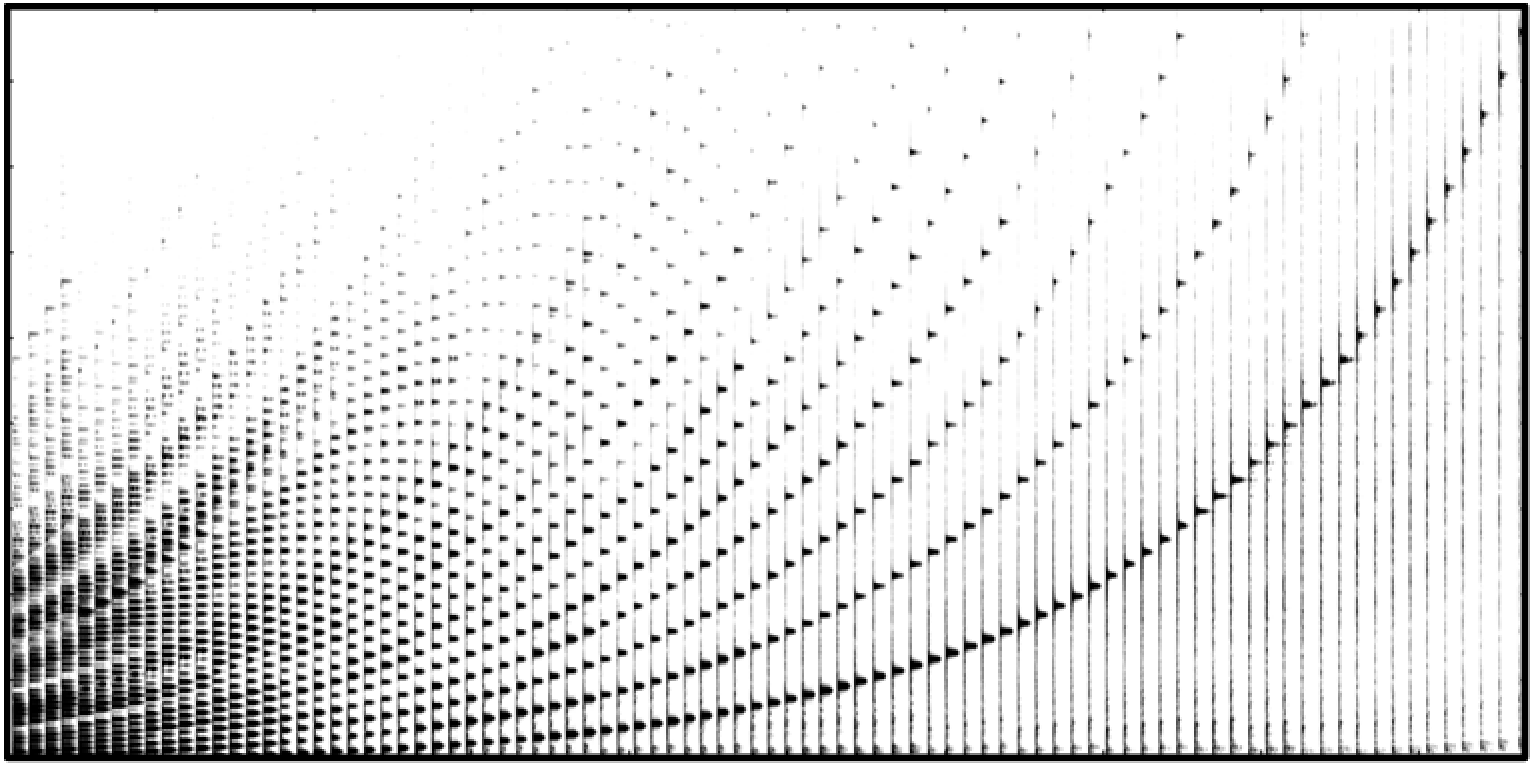
Chapter 2: Fourier Analysis of Signals



Important technical terminology is covered in Chapter 2. In particular, we approach the Fourier transform—which is perhaps the most fundamental tool in signal processing—from various perspectives. For the reader who is more interested in the musical aspects of the book, Section 2.1 provides a summary of the most important facts on the Fourier transform. In particular, the notion of a spectrogram, which yields a time–frequency representation of an audio signal, is introduced. The remainder of the chapter treats the Fourier transform in greater mathematical depth and also includes the fast Fourier transform (FFT)—an algorithm of great beauty and high practical relevance.

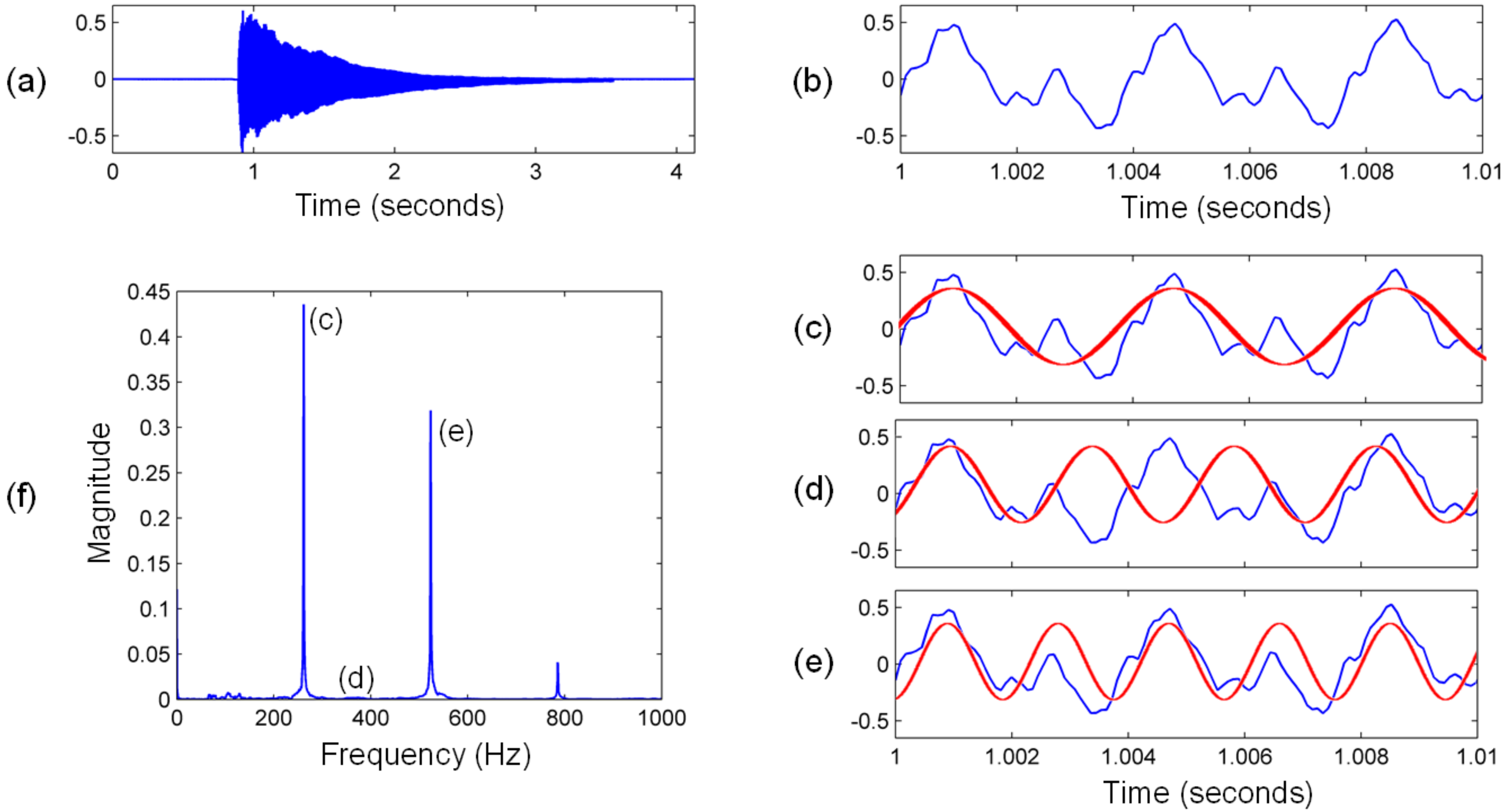
2 Fourier Analysis of Signals

Teaser



2.1 The Fourier Transform in a Nutshell

Fig. 2.1



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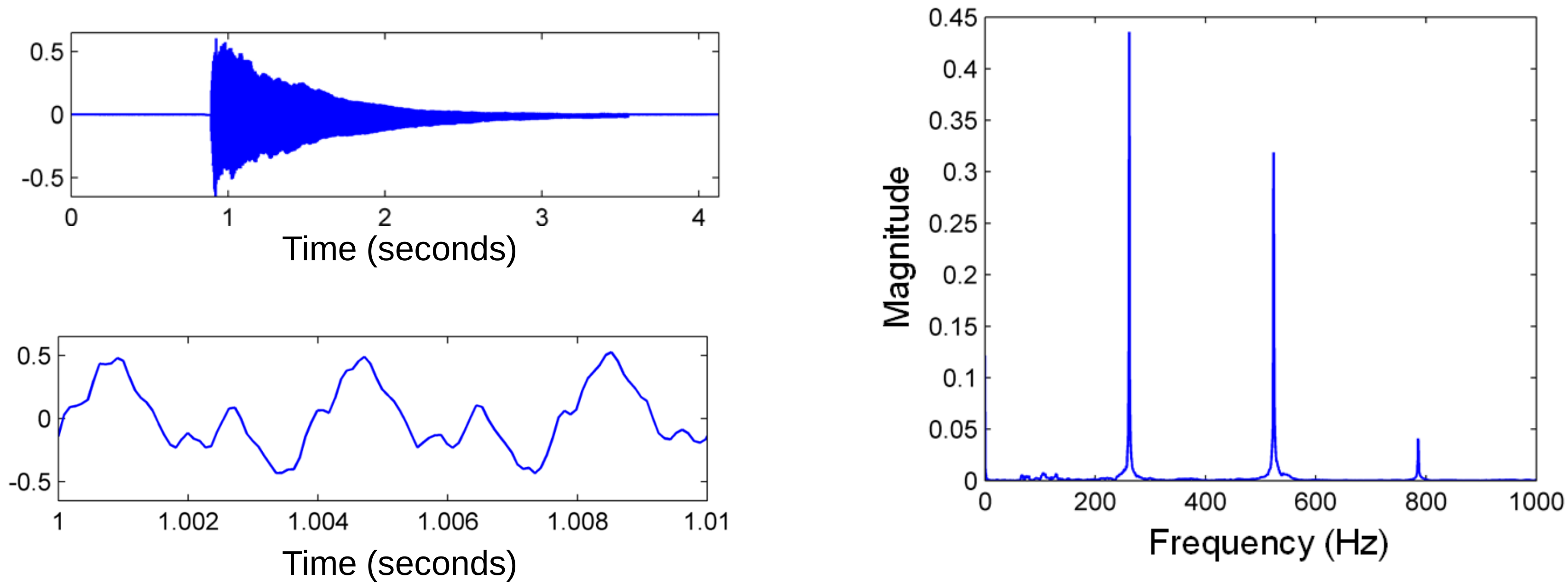

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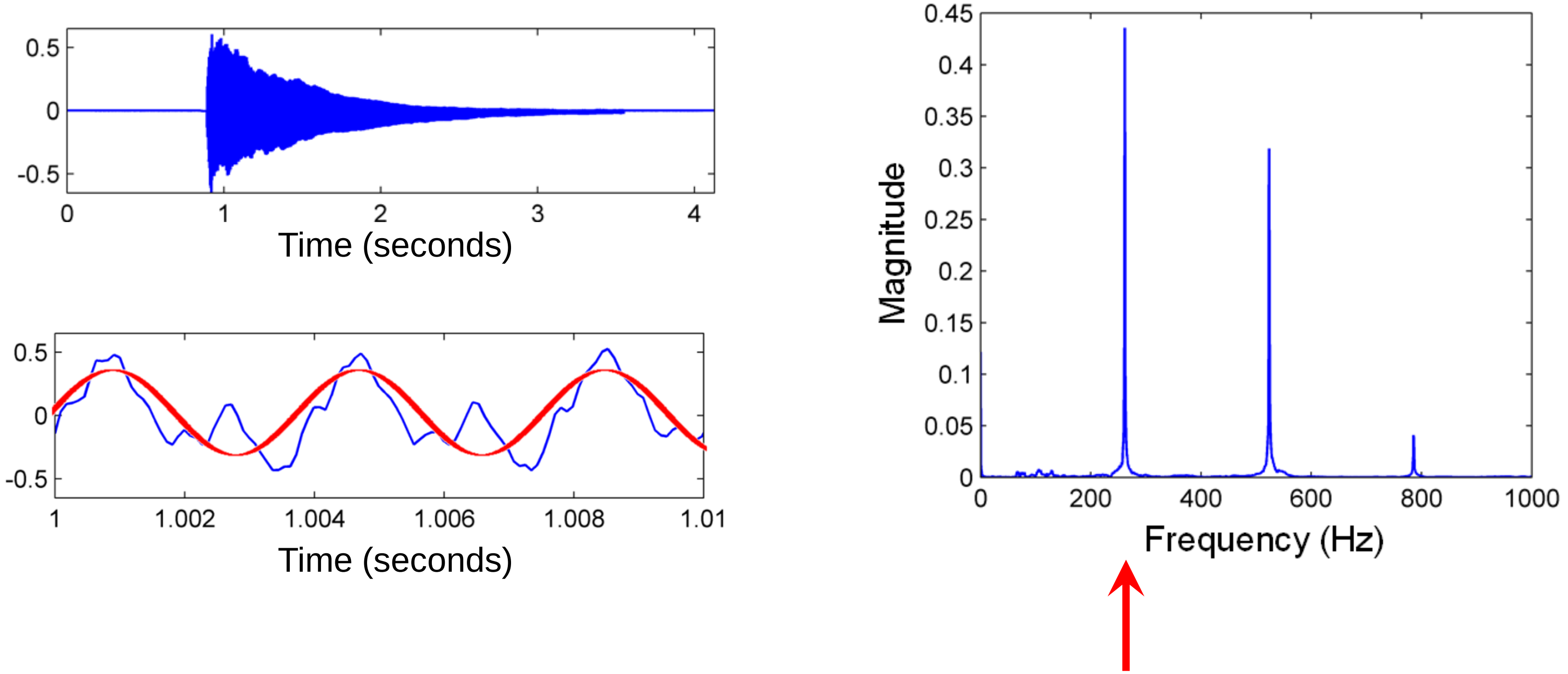

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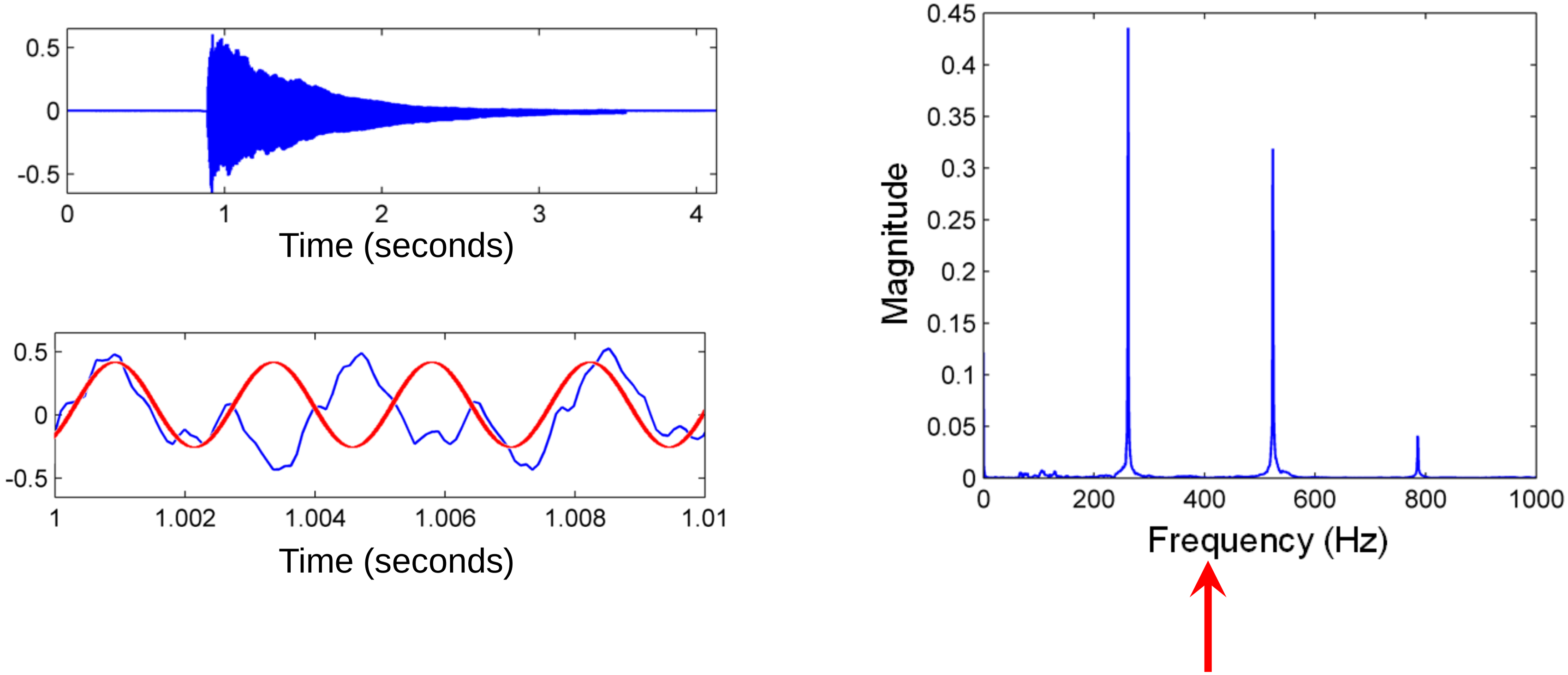

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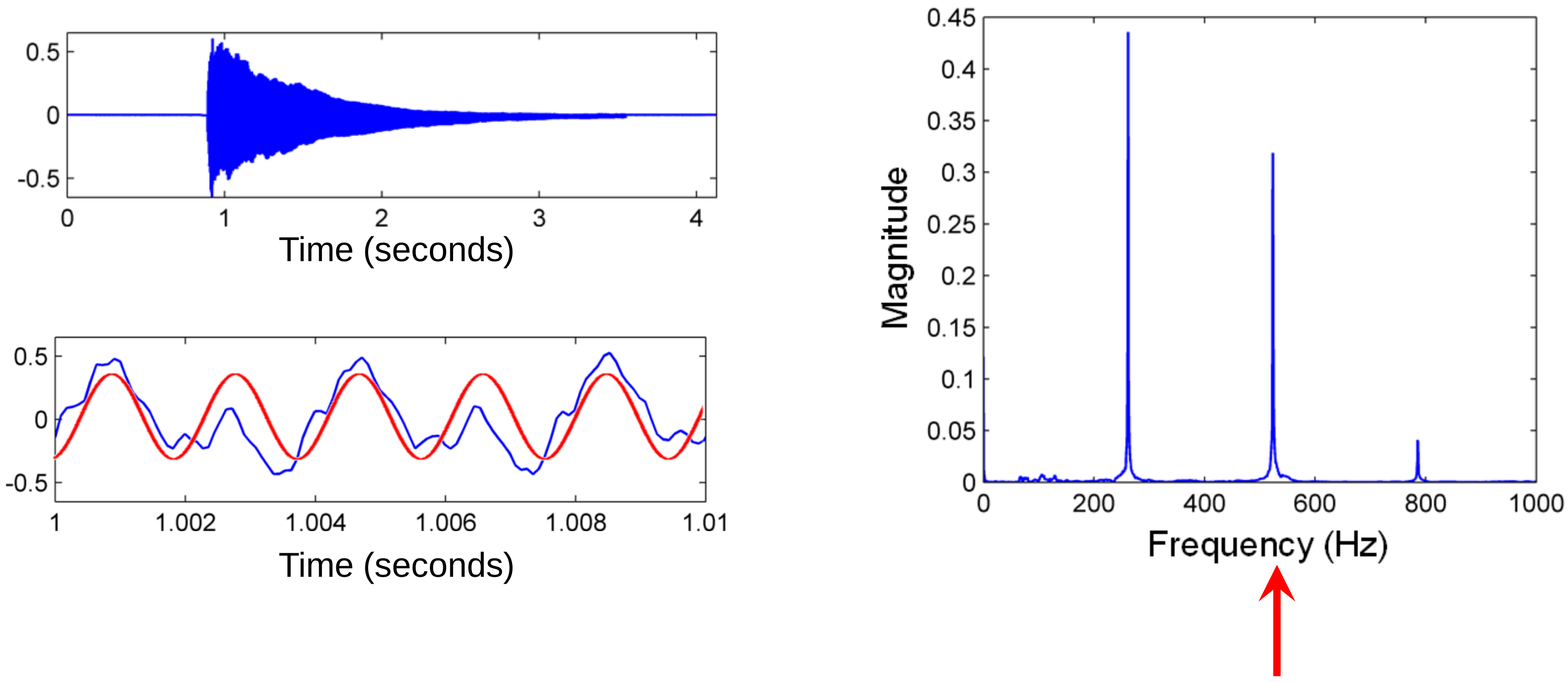

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Fig. 2.2

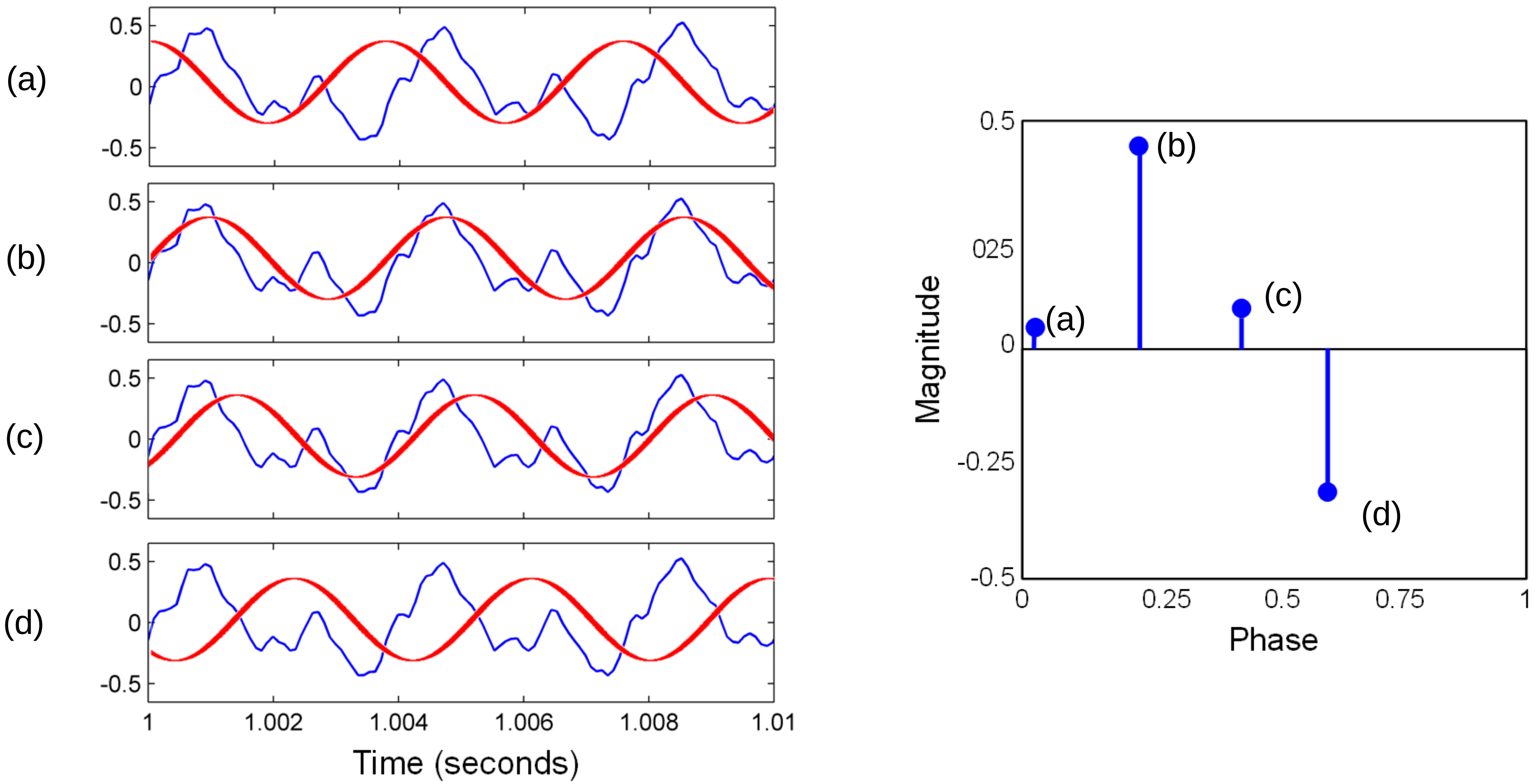

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Fig. 2.3

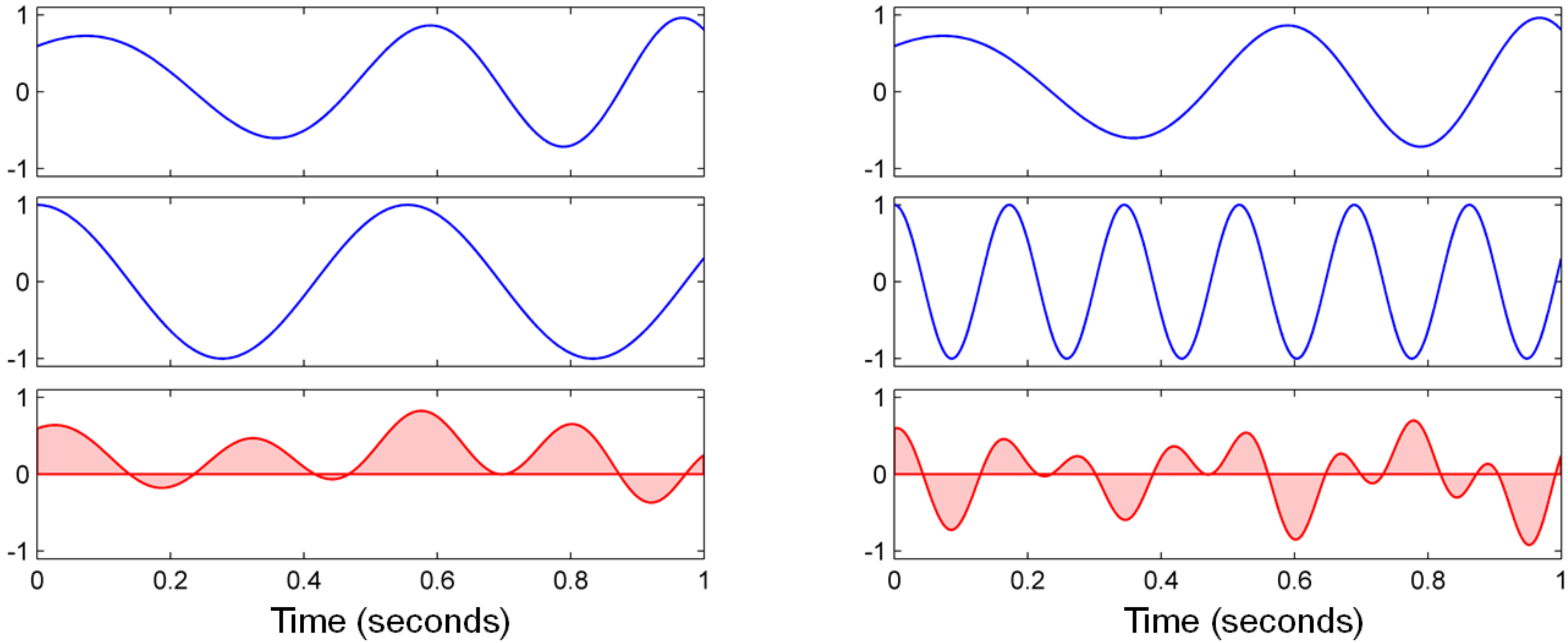

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Fig. 2.4

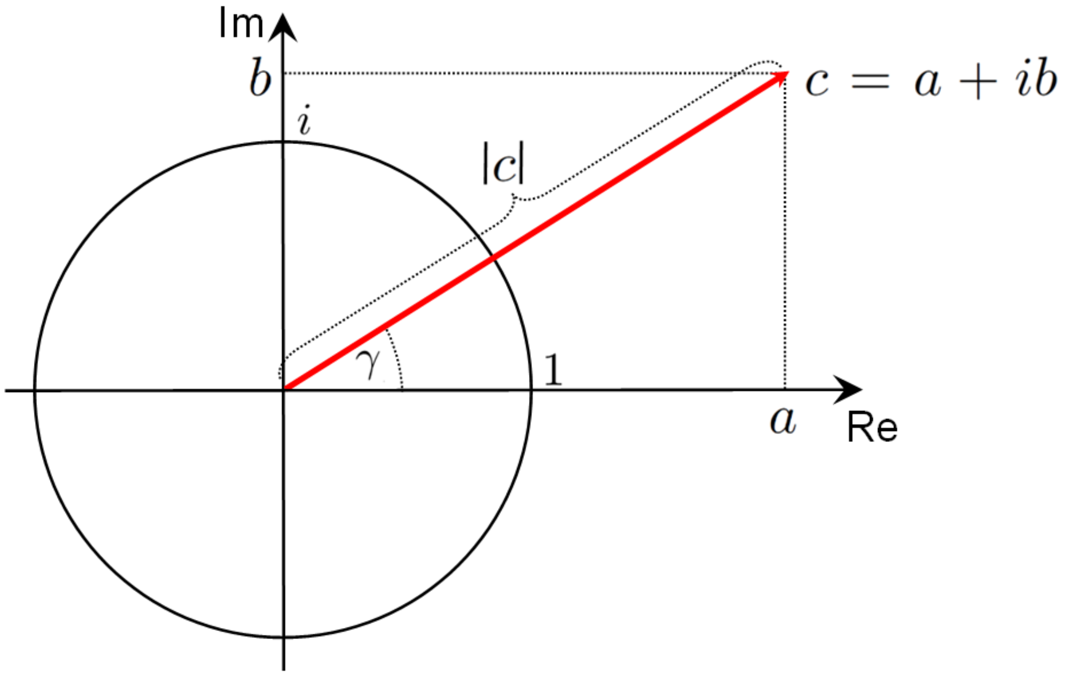

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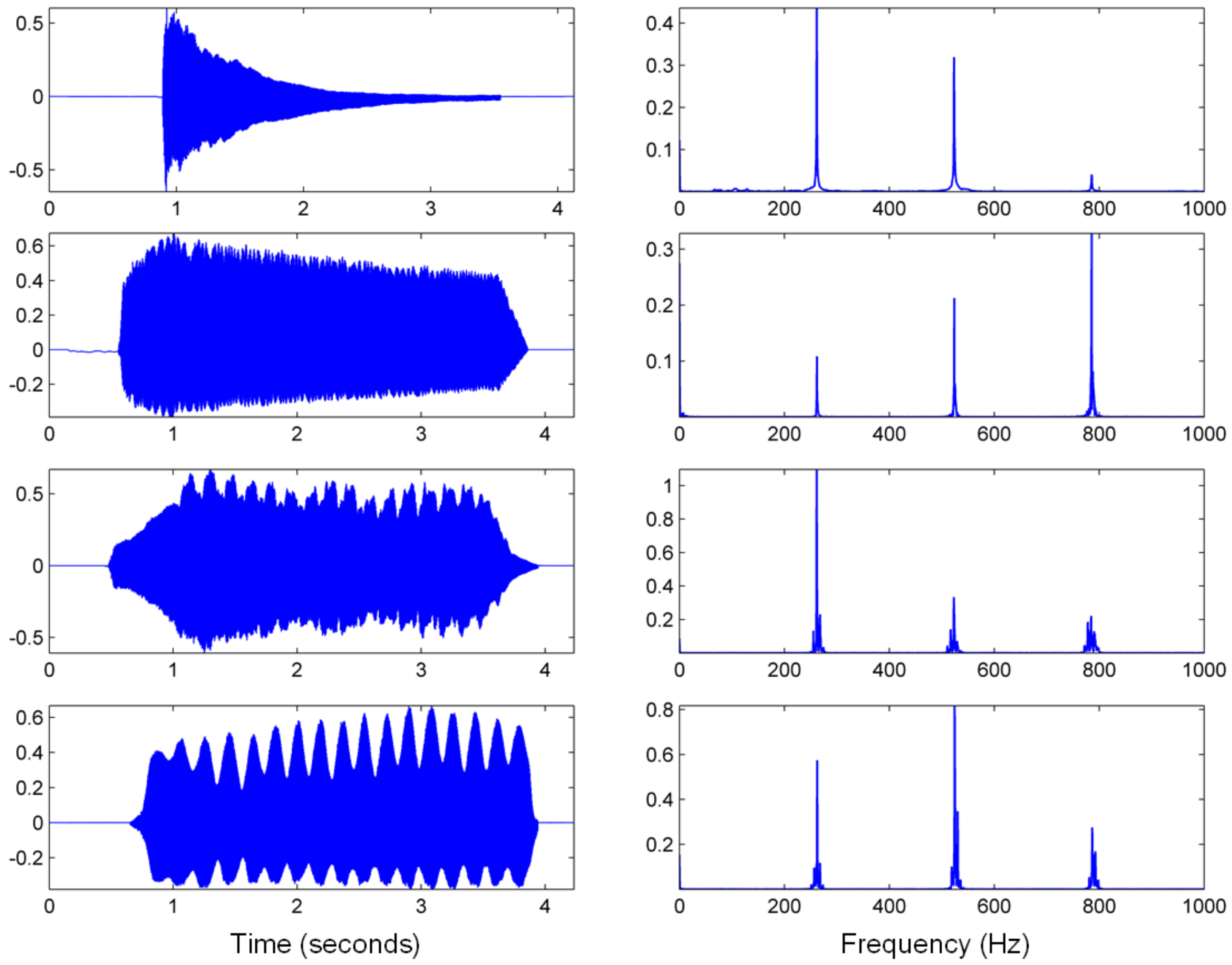
Fig. 2.5



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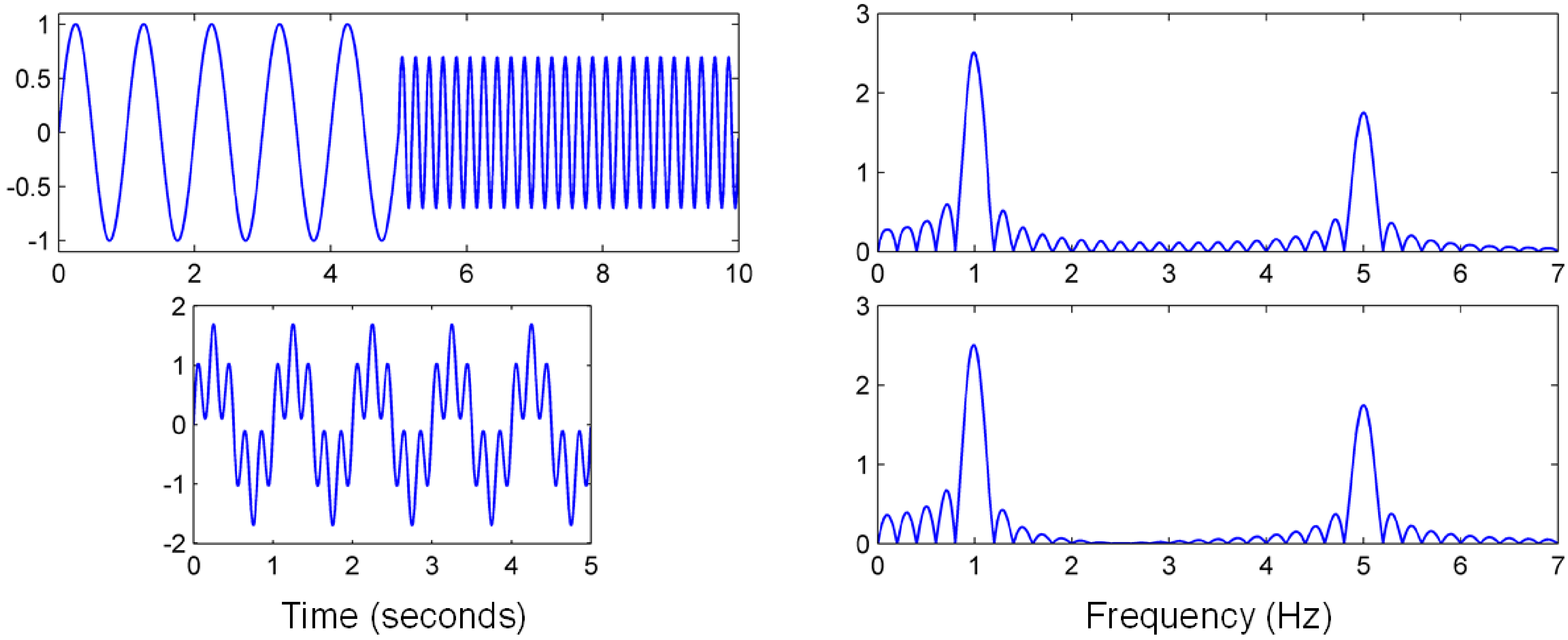

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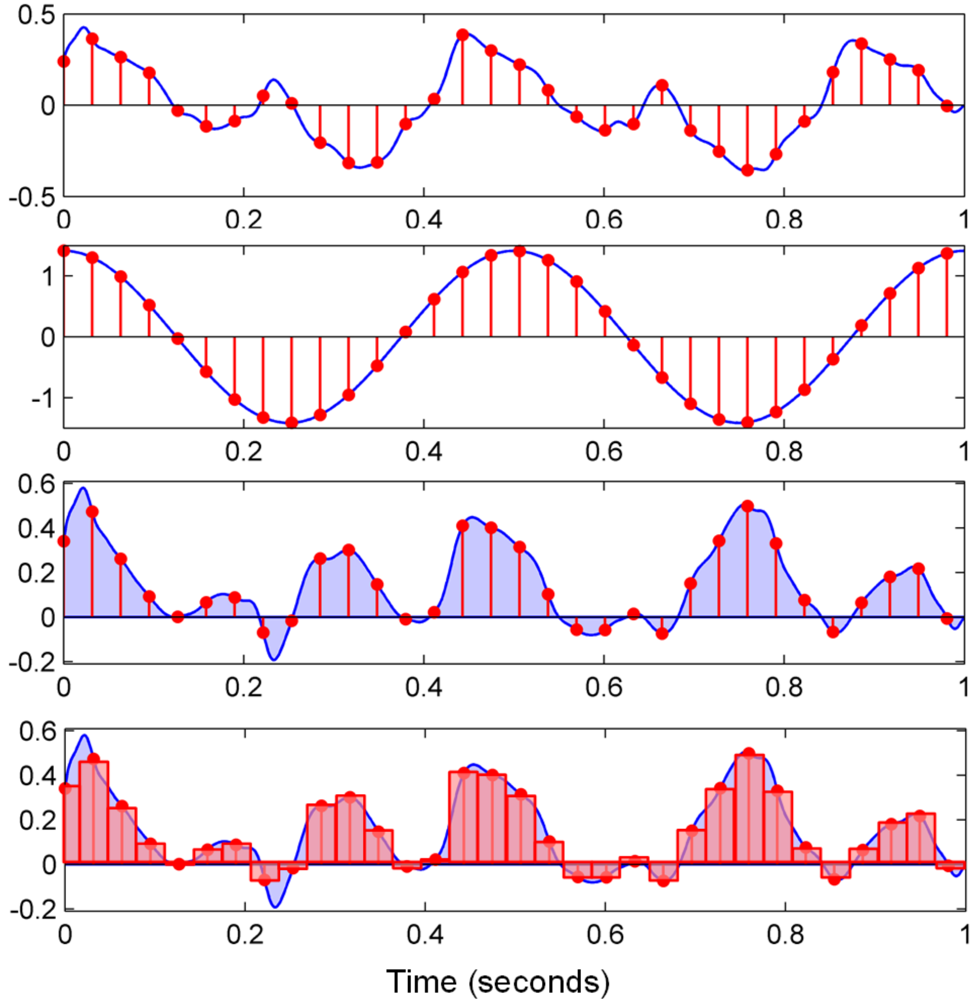

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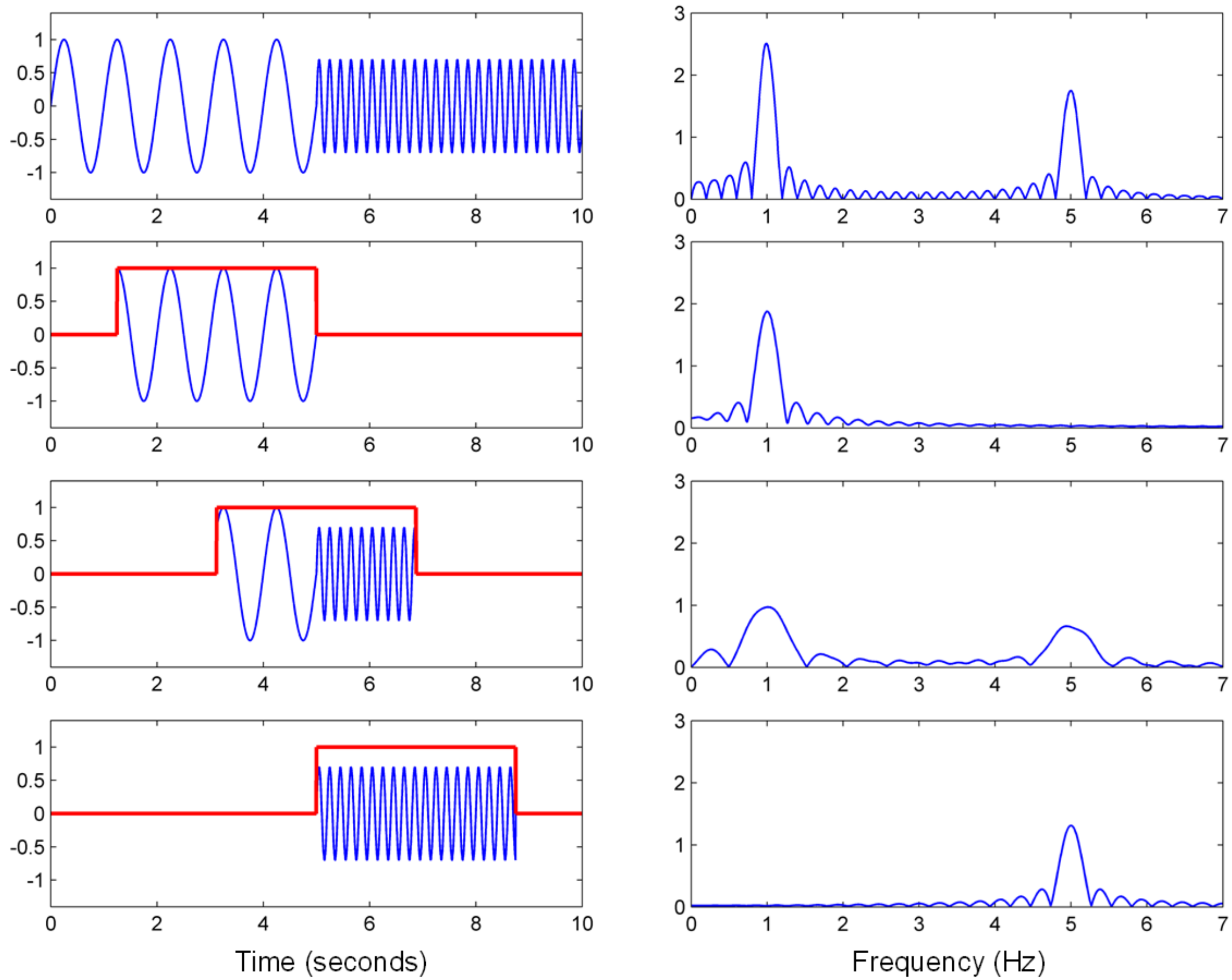
Fig. 2.8



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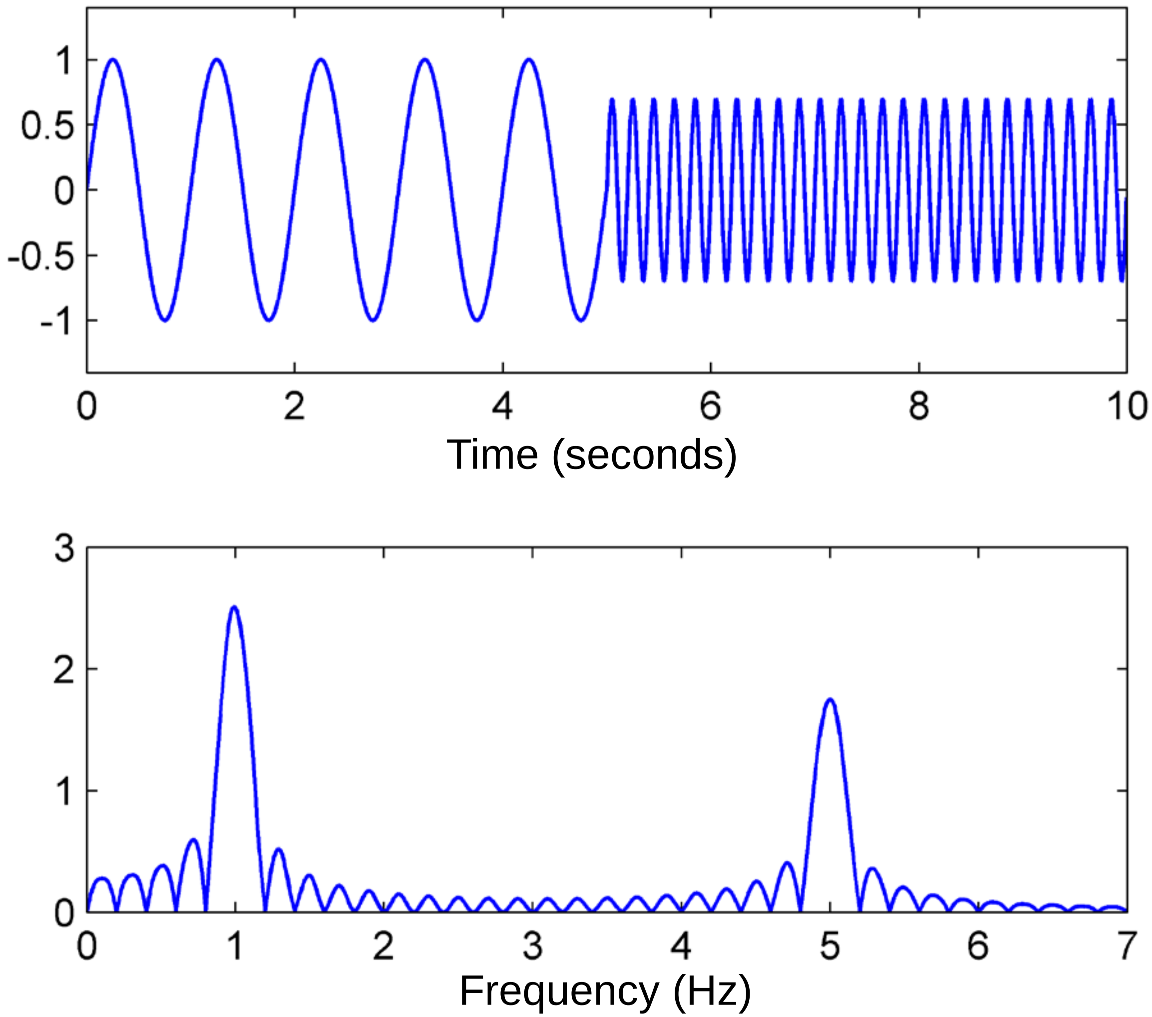

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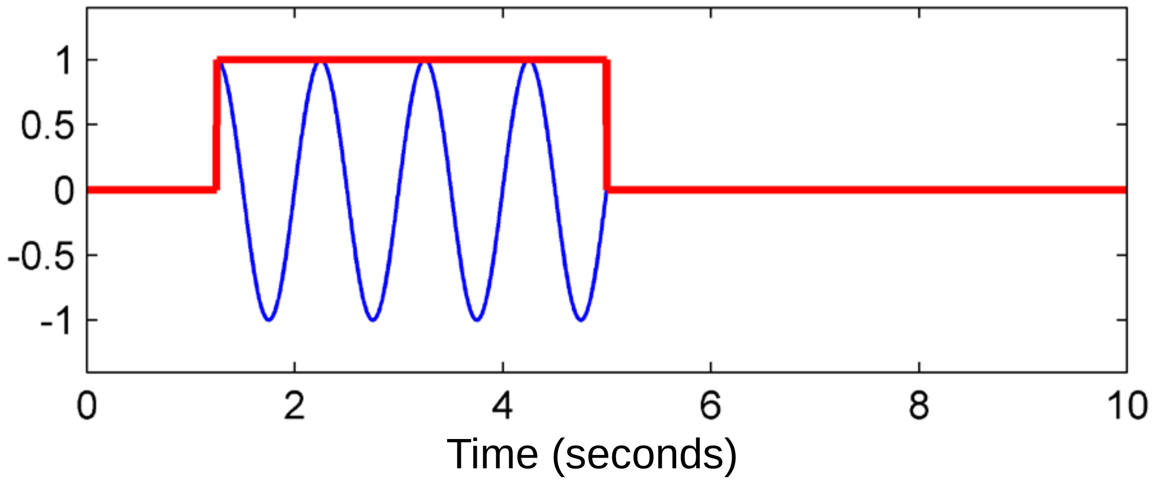

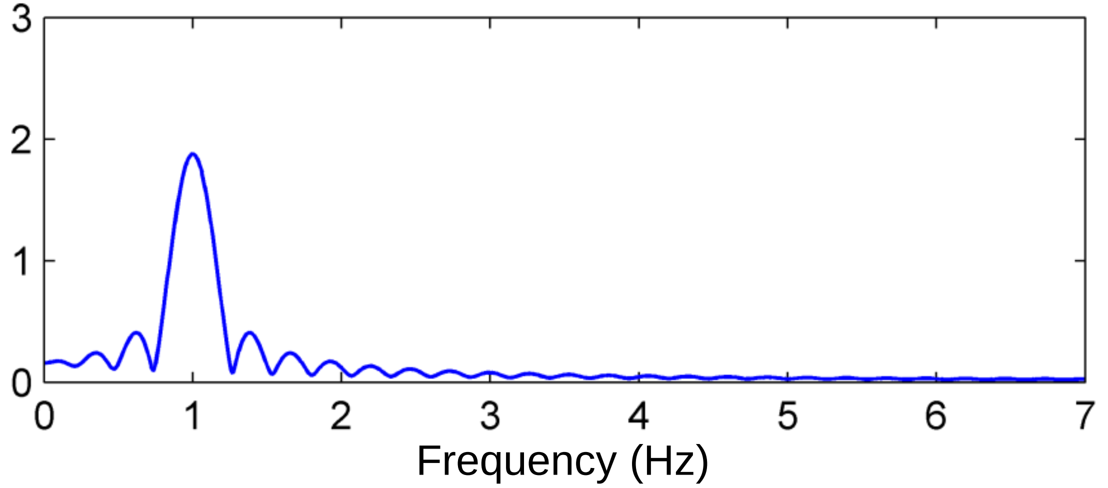

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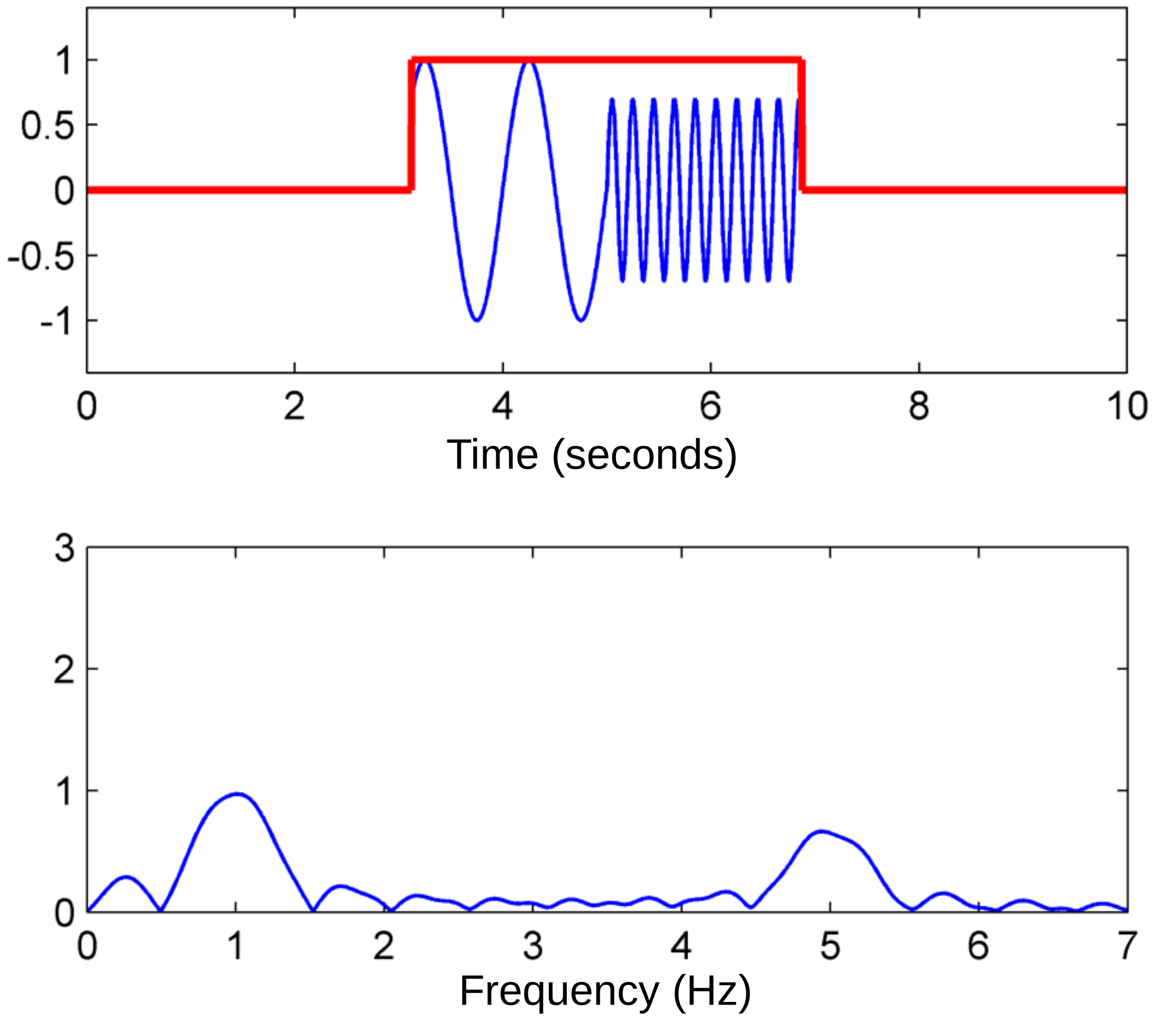

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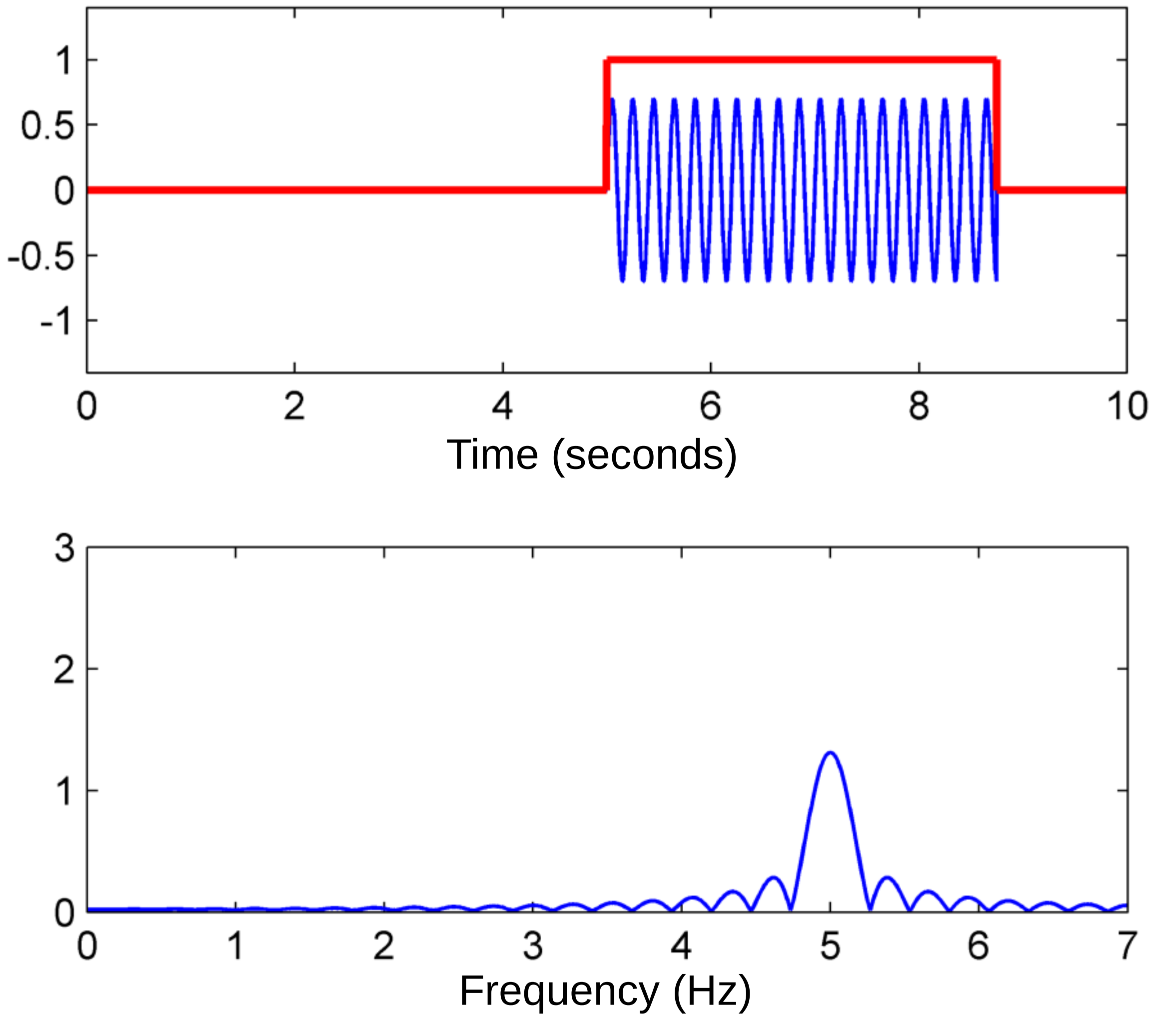

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Fig. 2.9

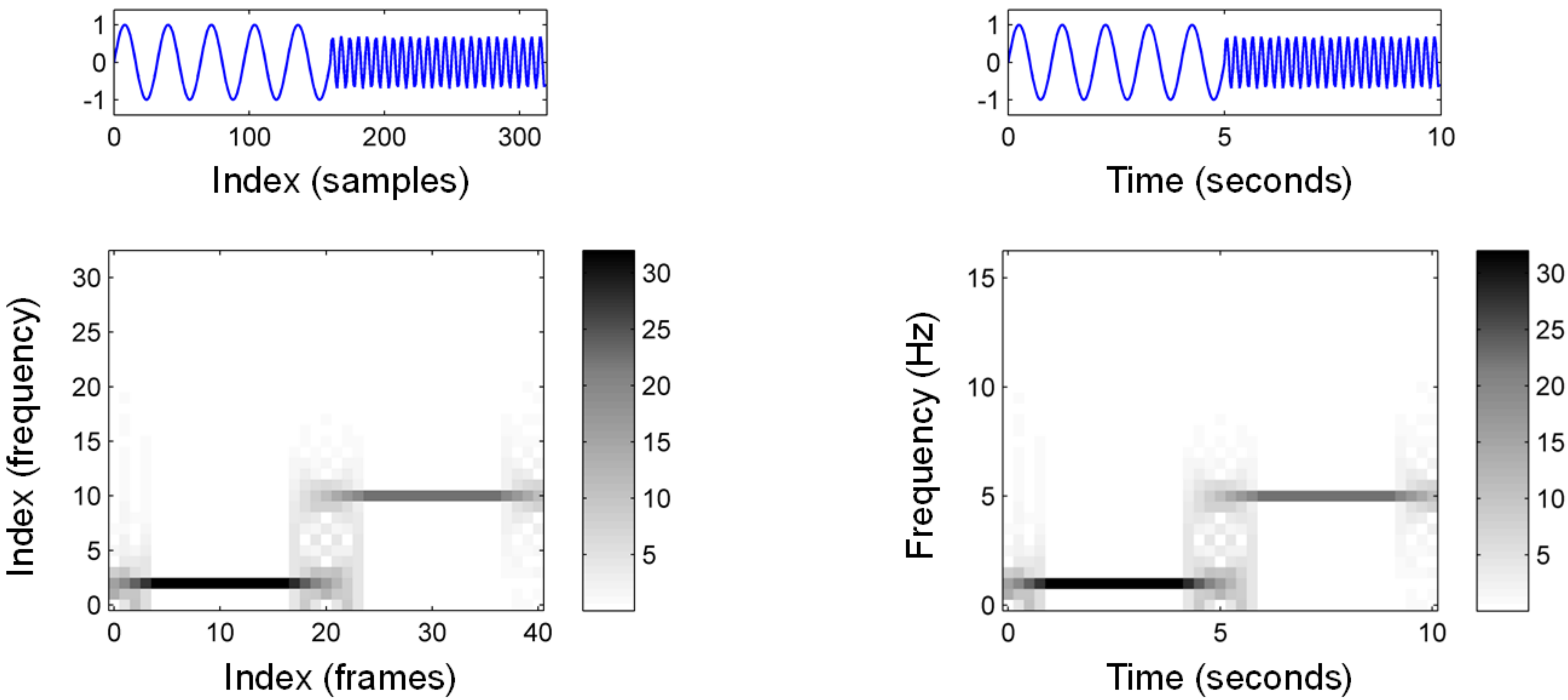

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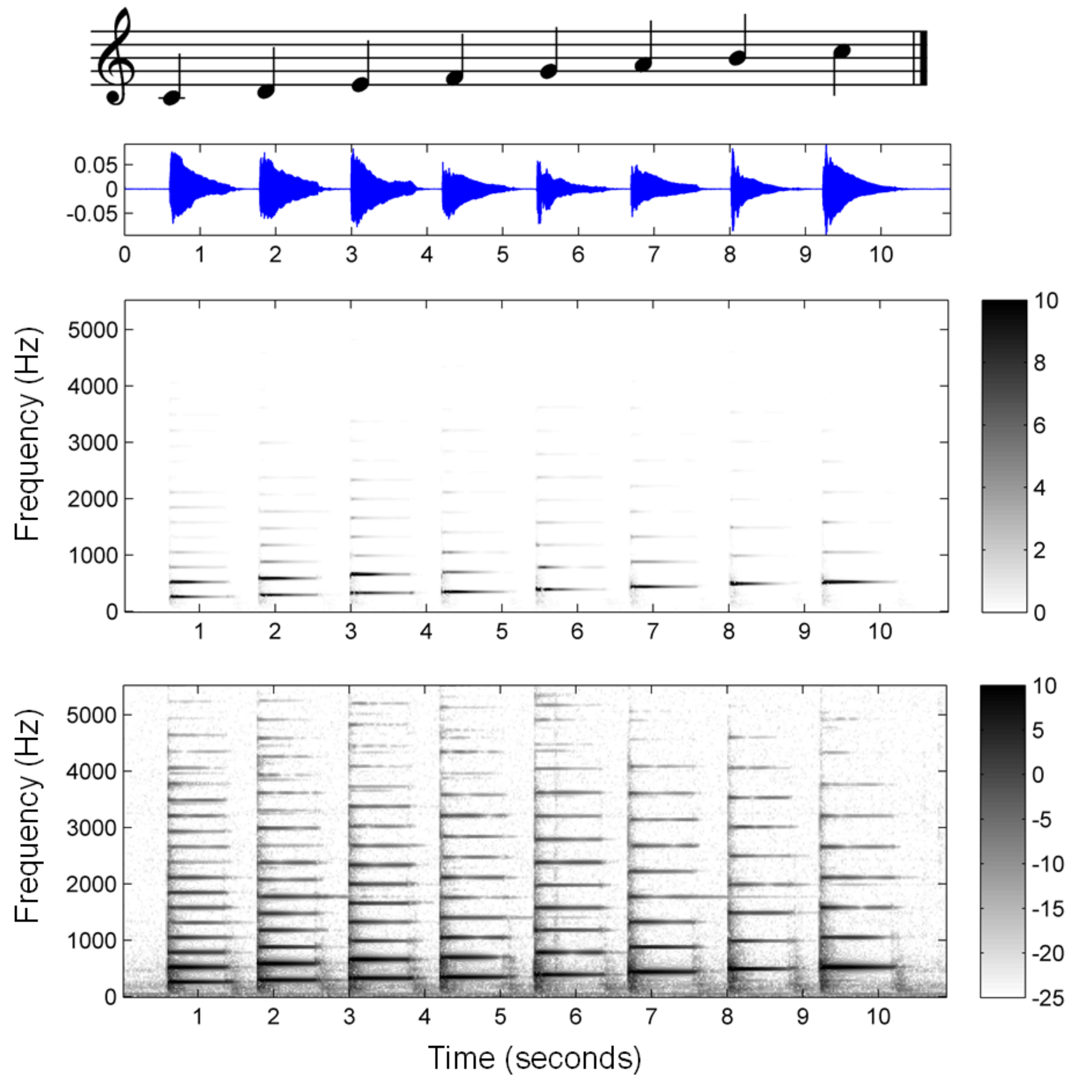
Fig. 2.10



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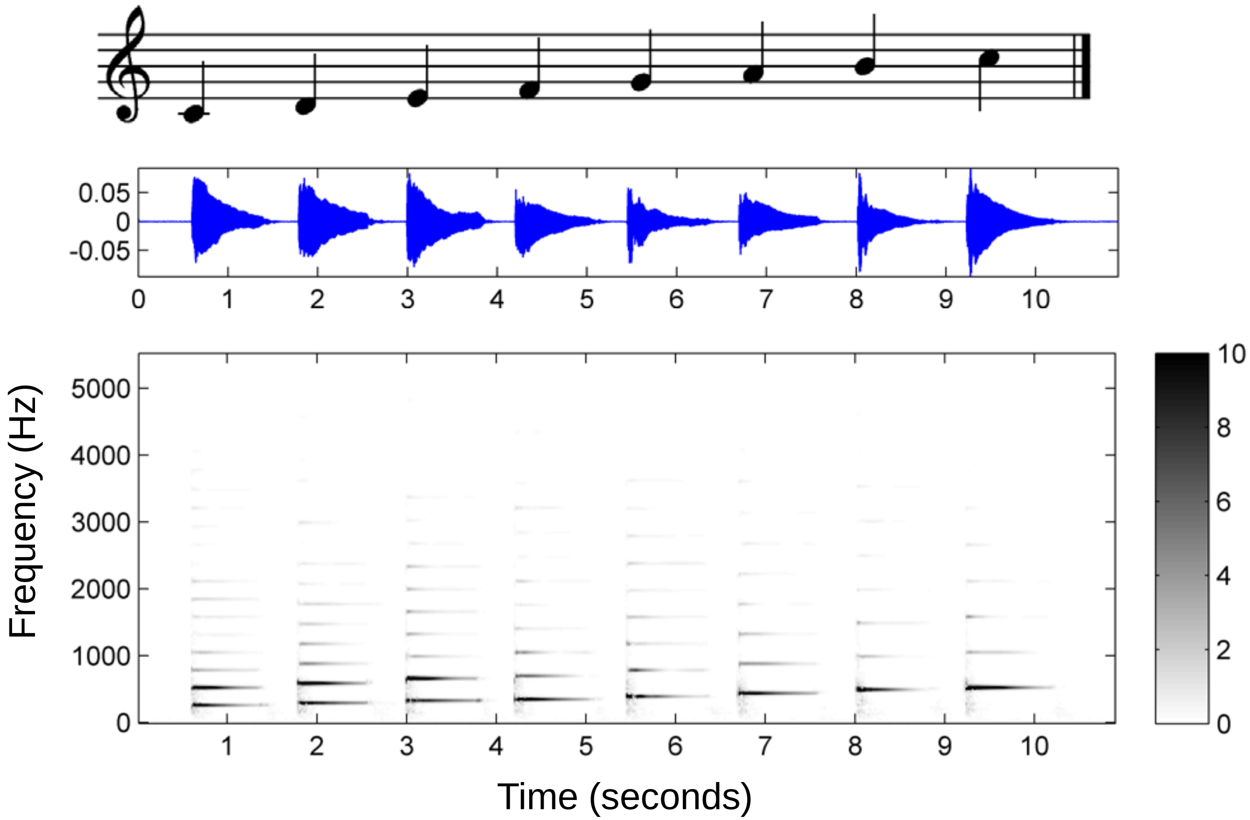

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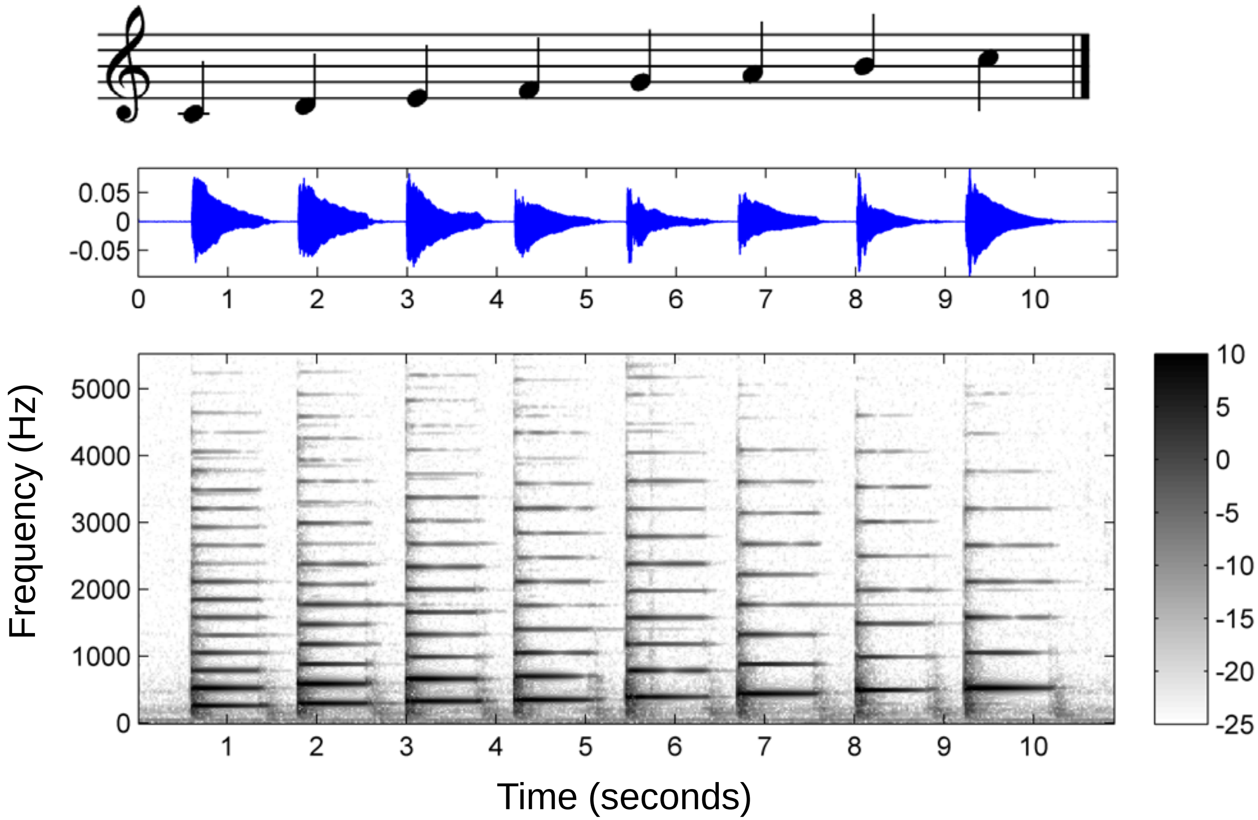
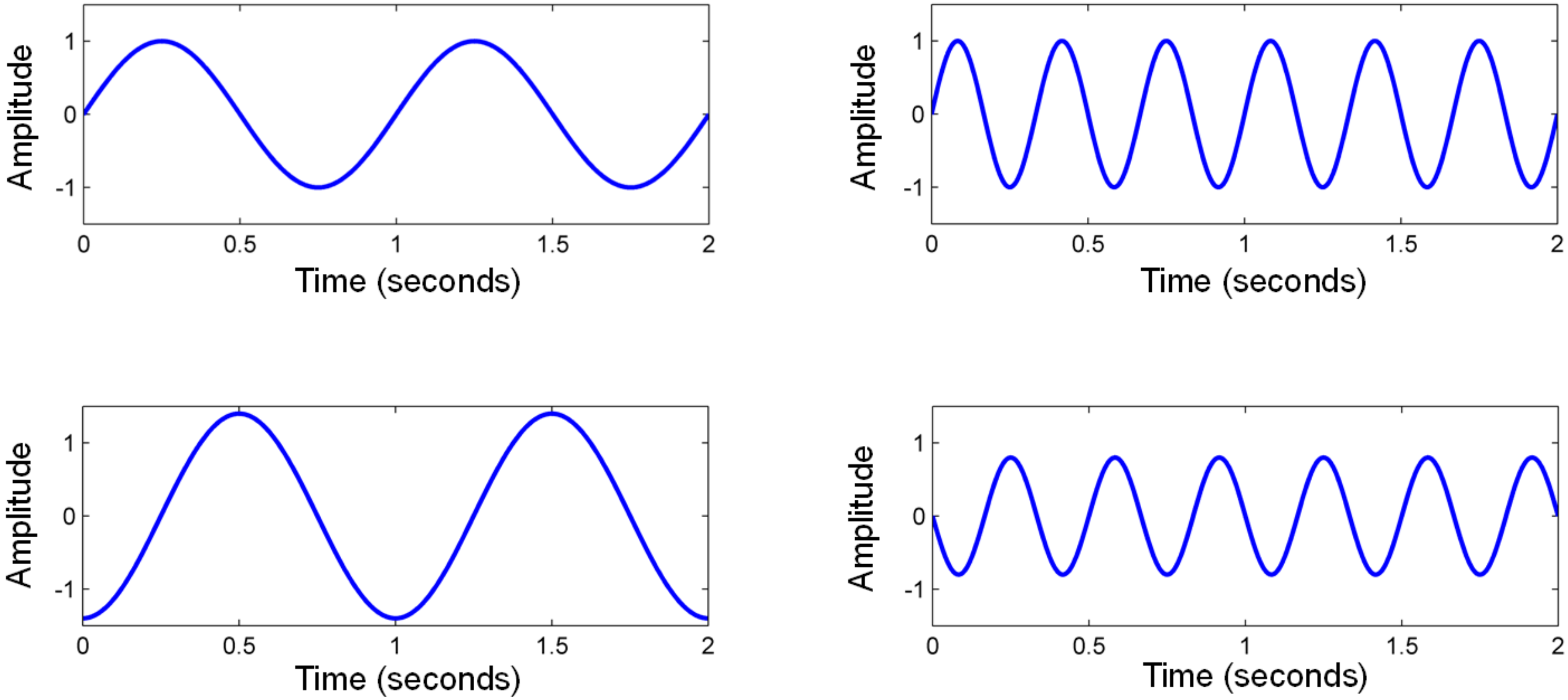


2.2 Signals and Signal Spaces

Fig. 2.11



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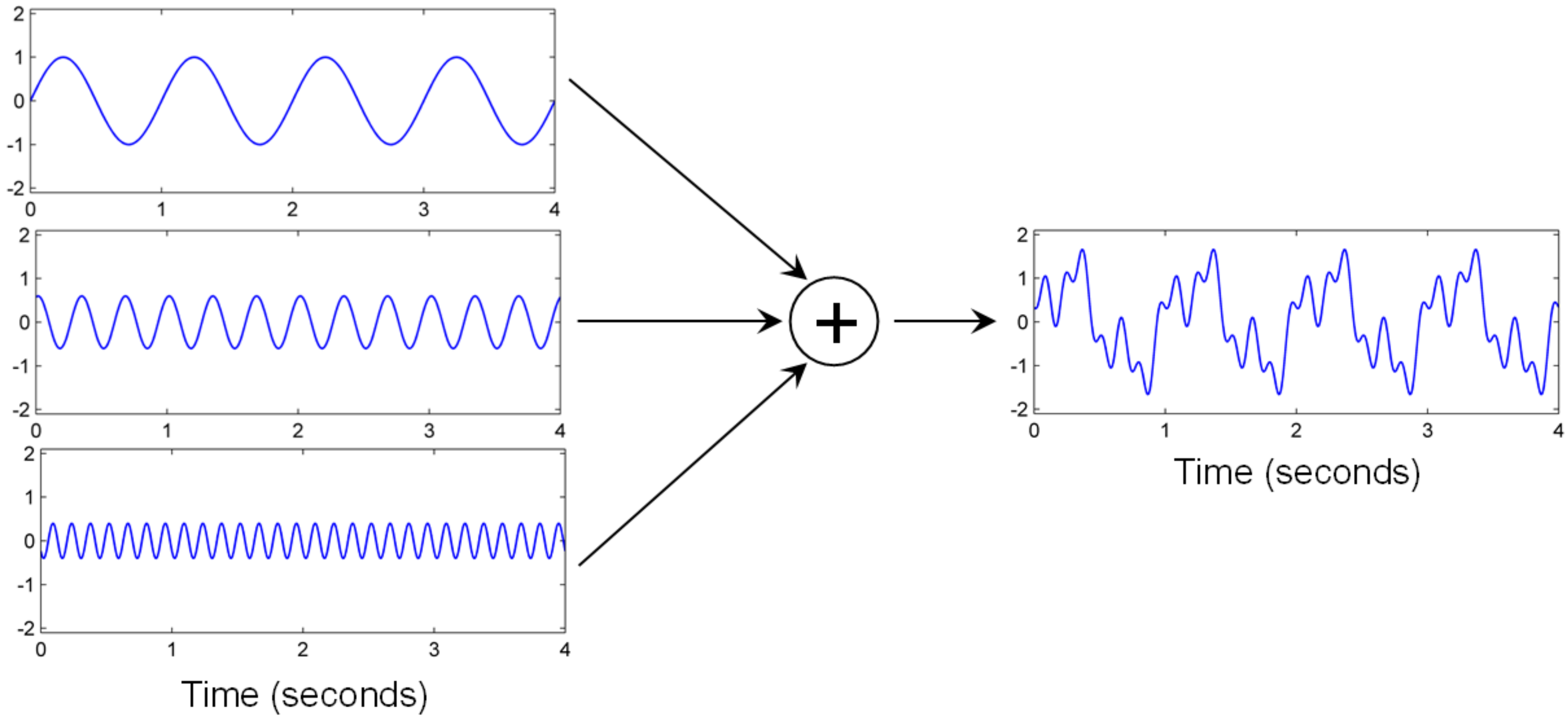

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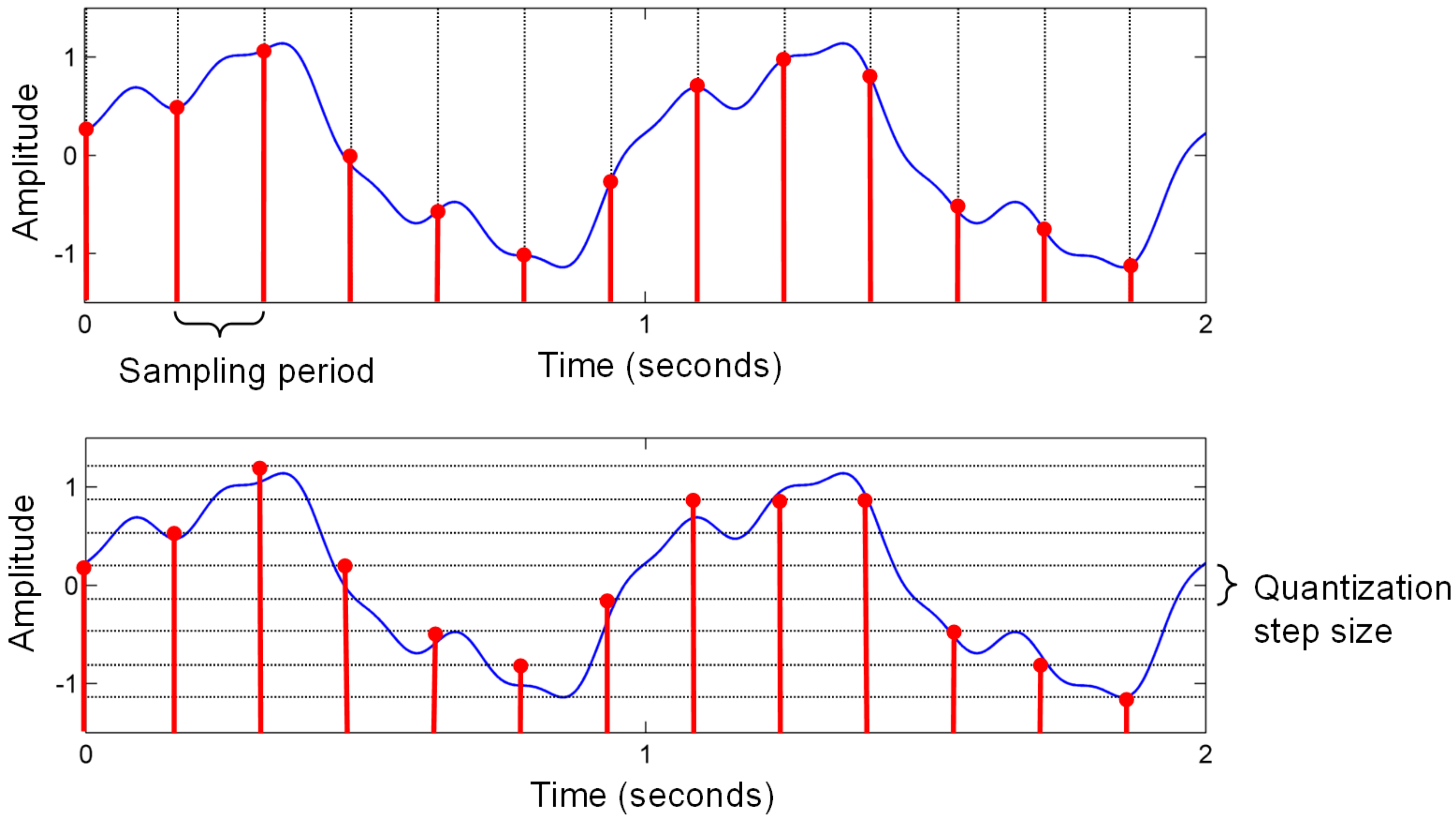

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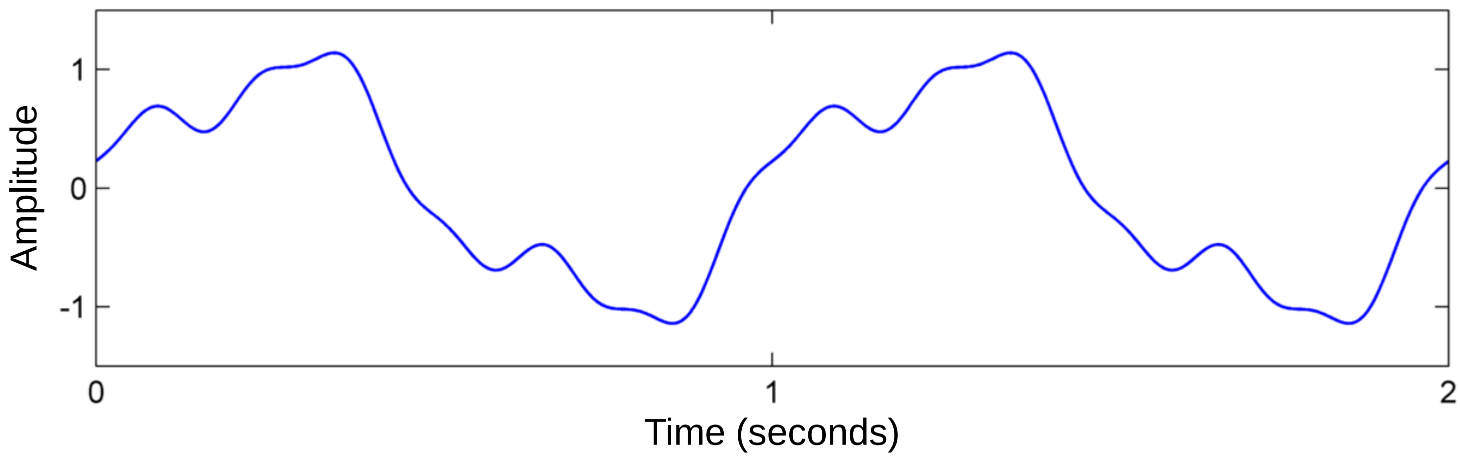

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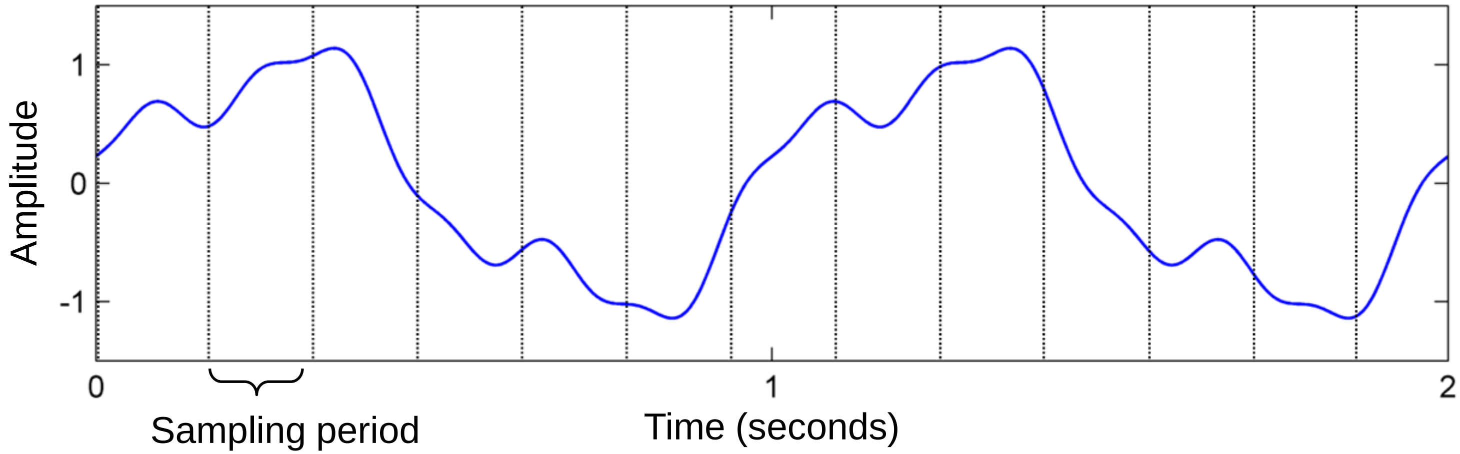

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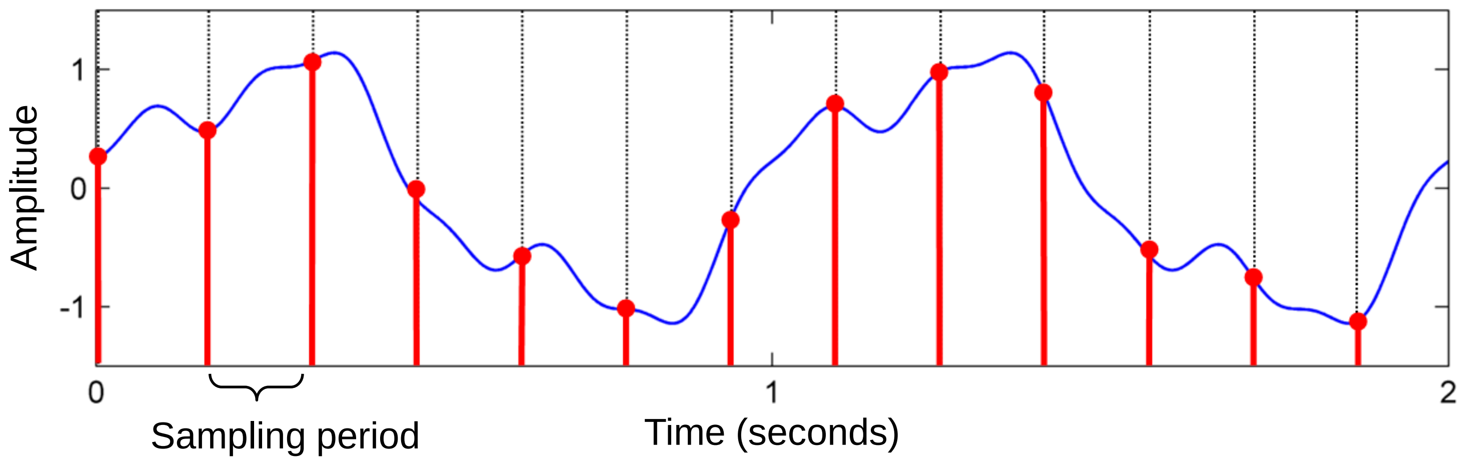

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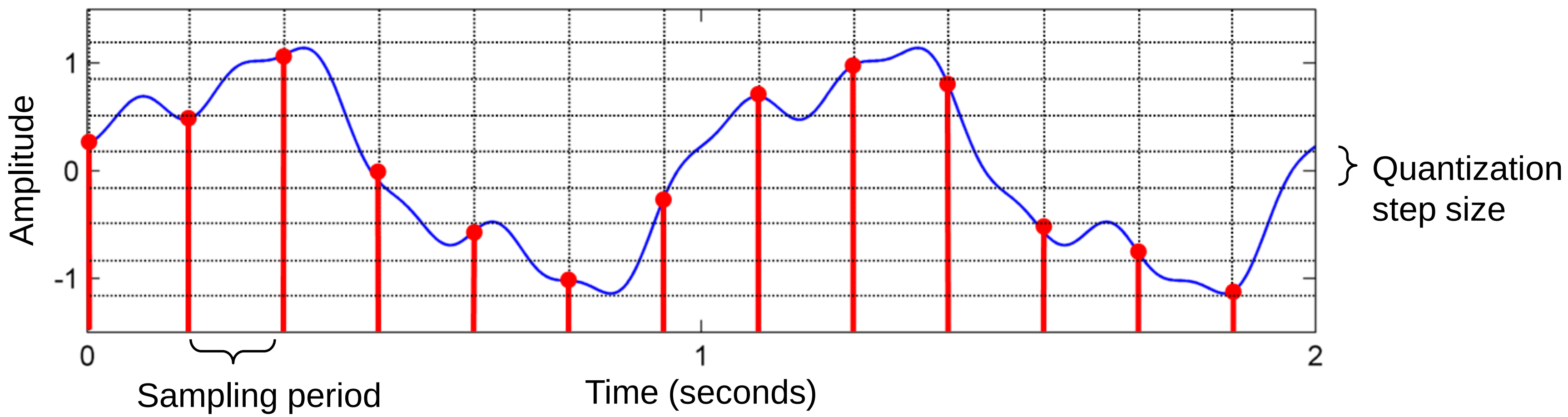

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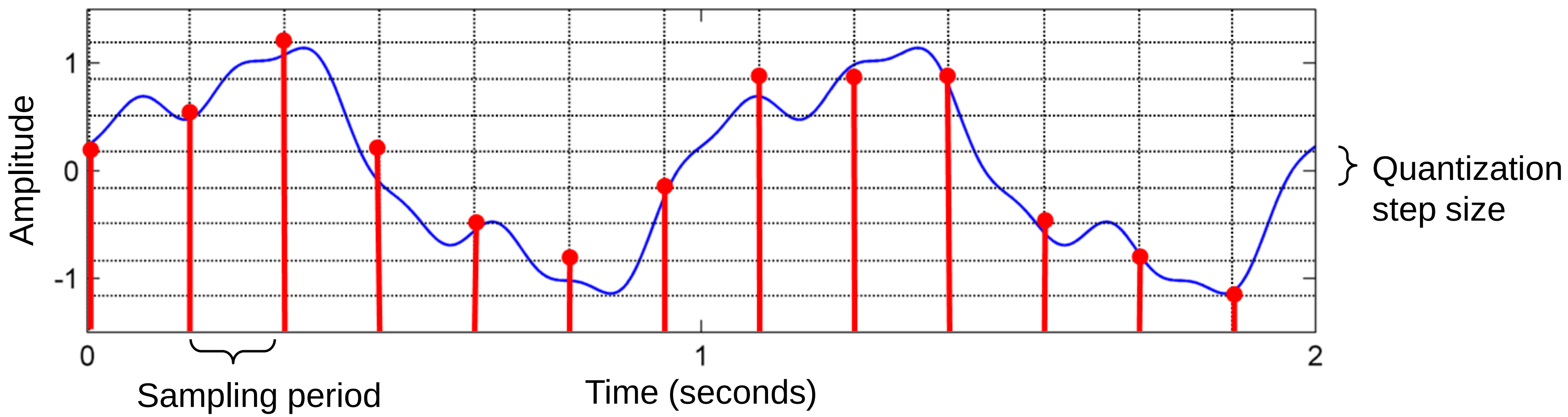

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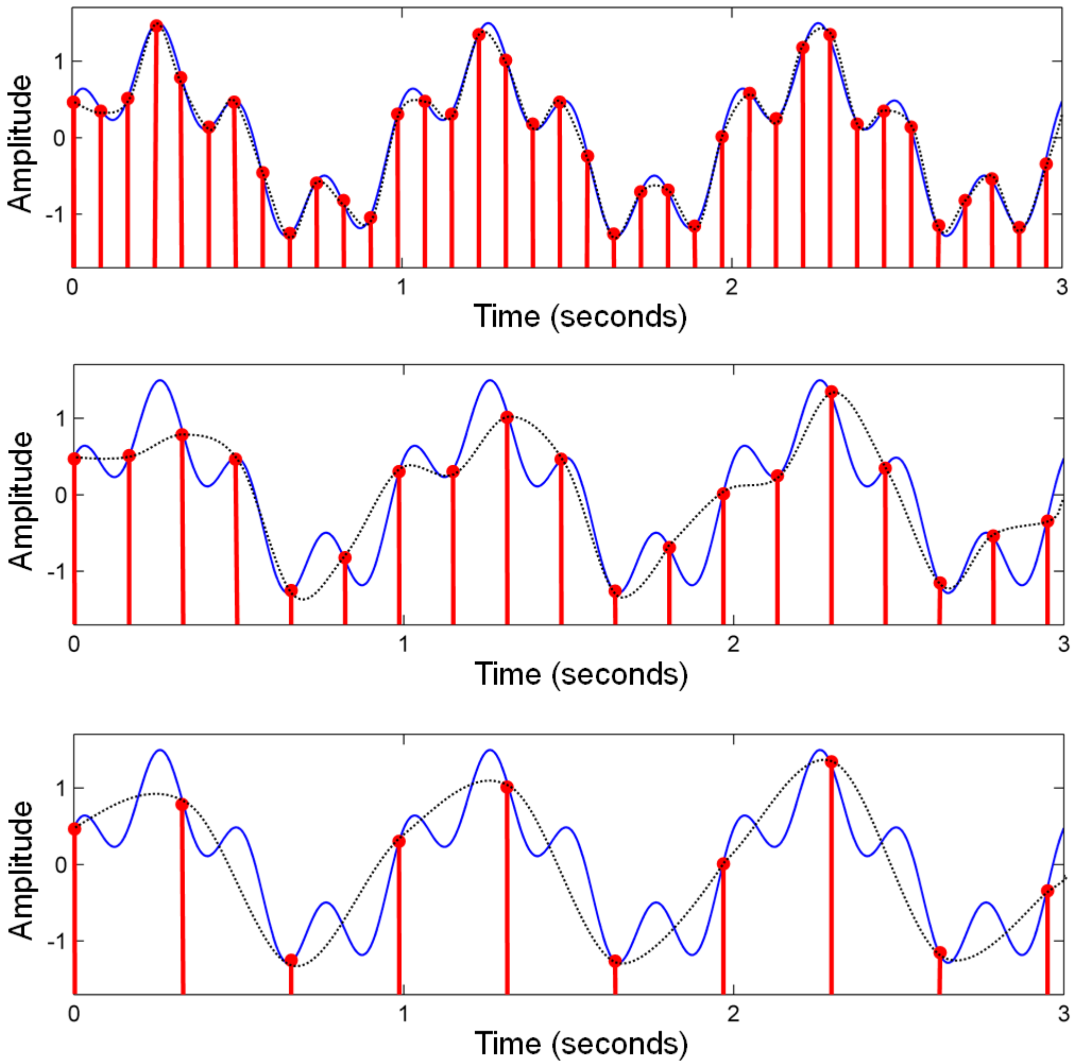

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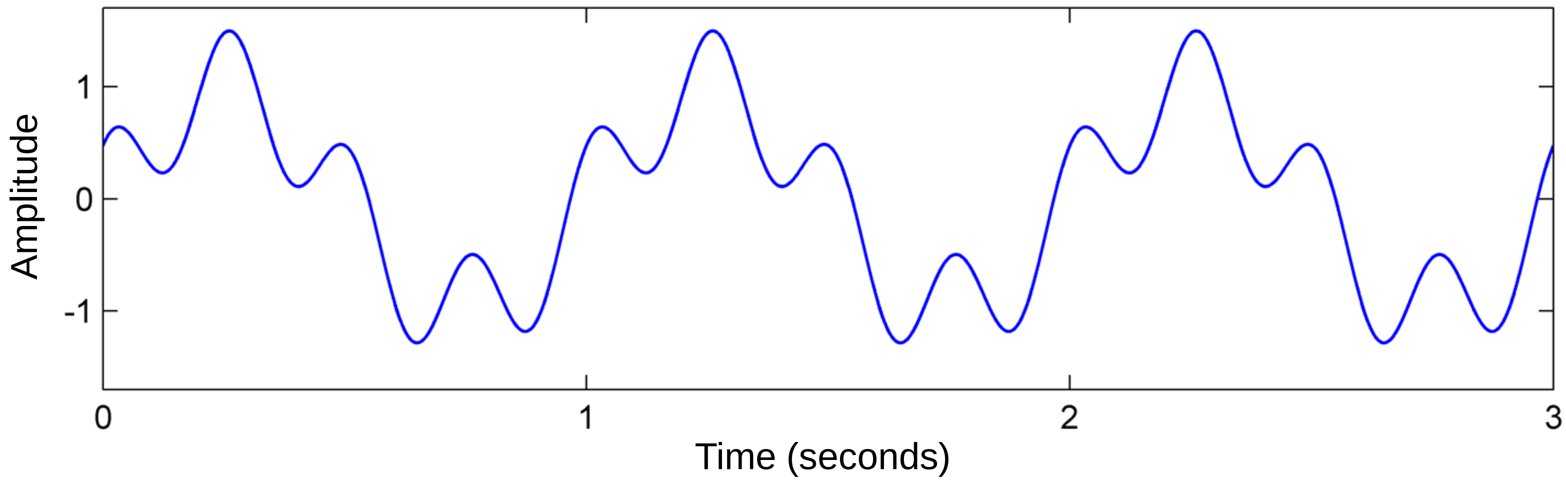

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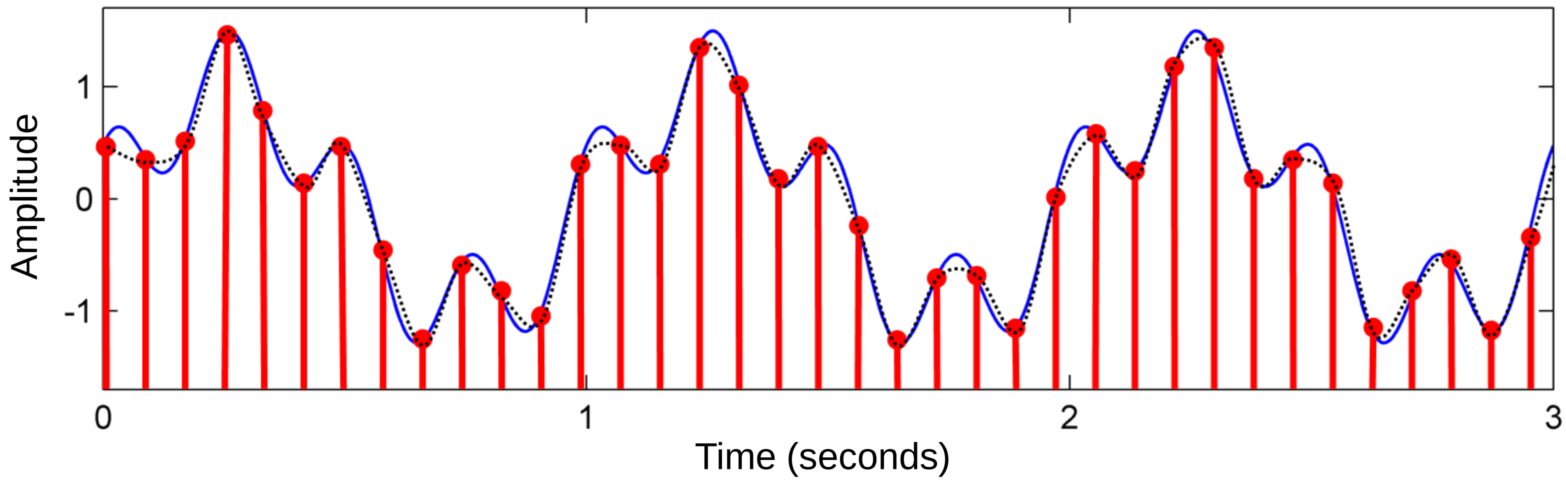

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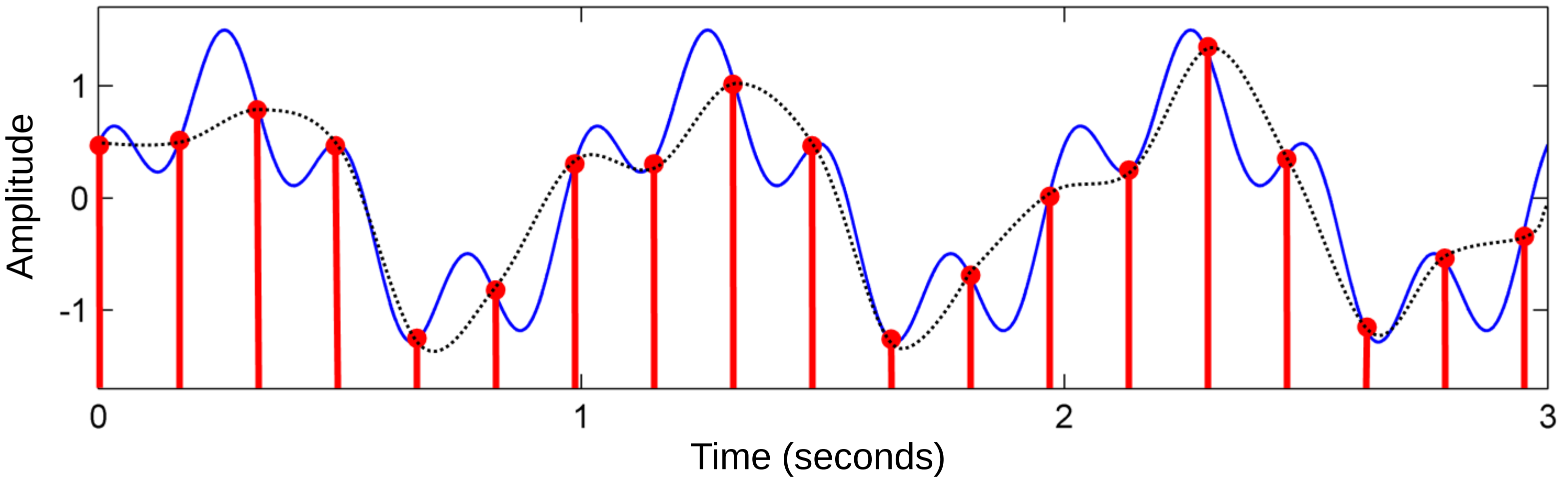

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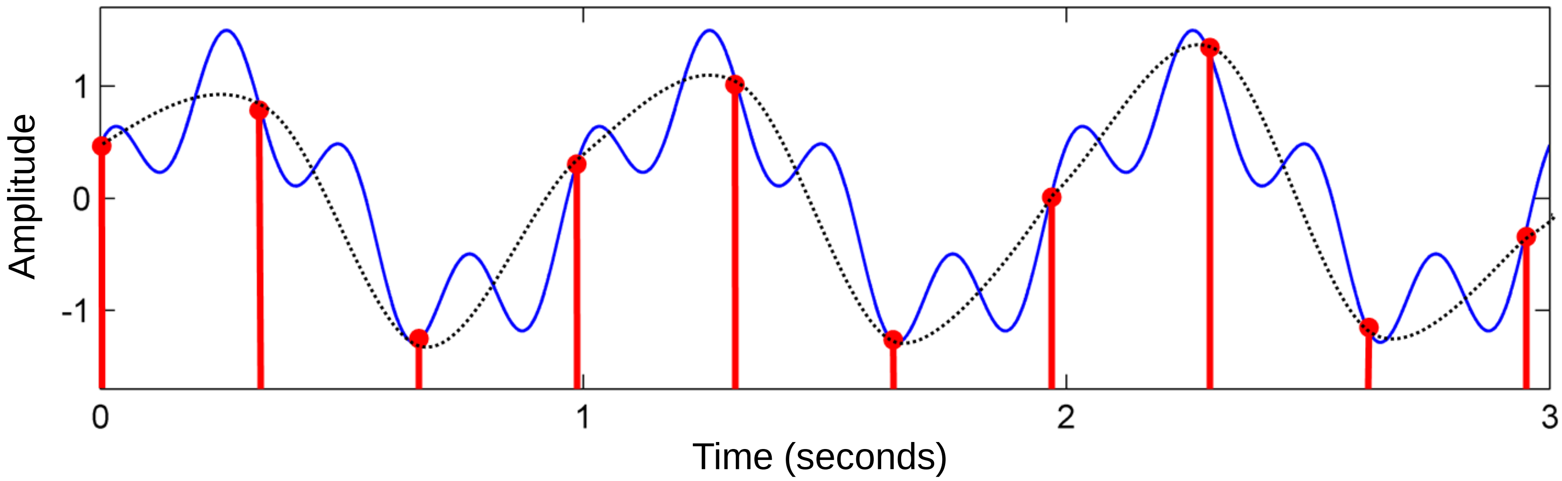

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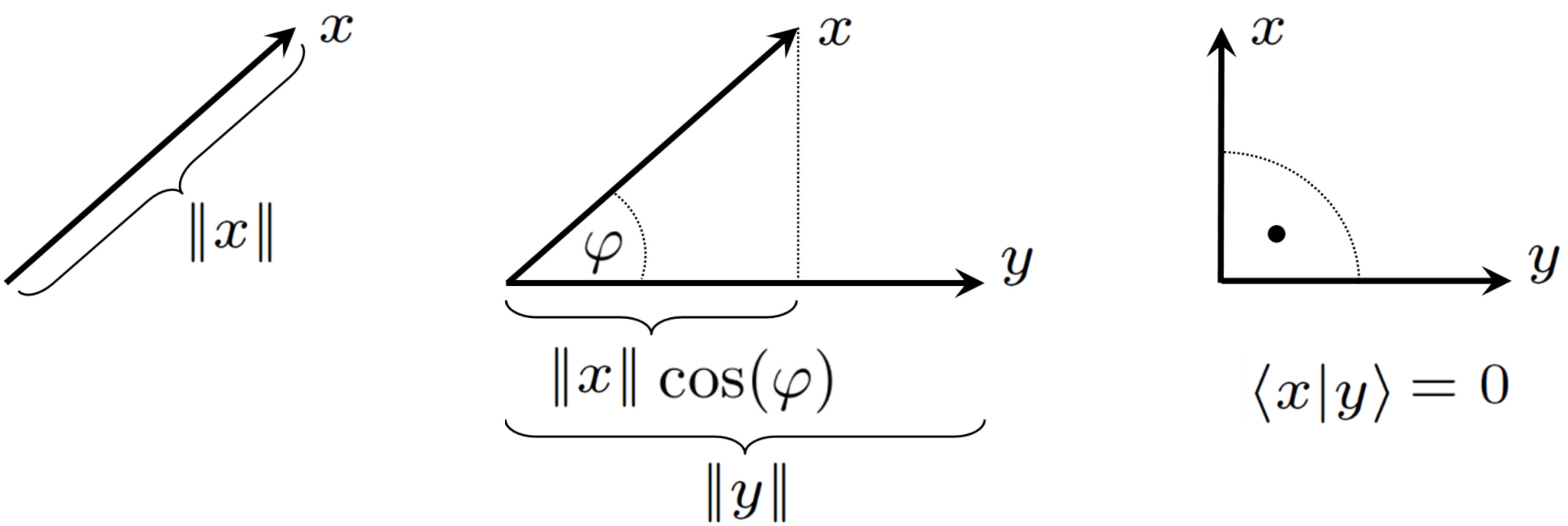
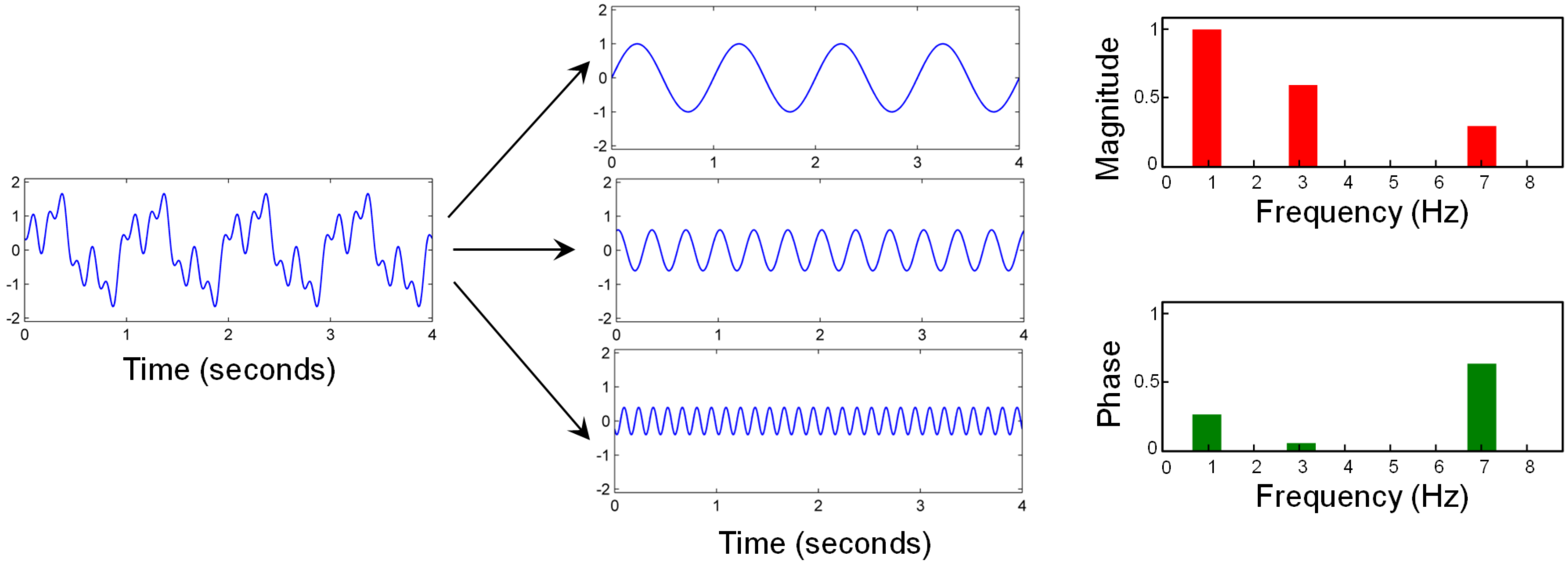


2.3 Fourier Transform

Fig. 2.16



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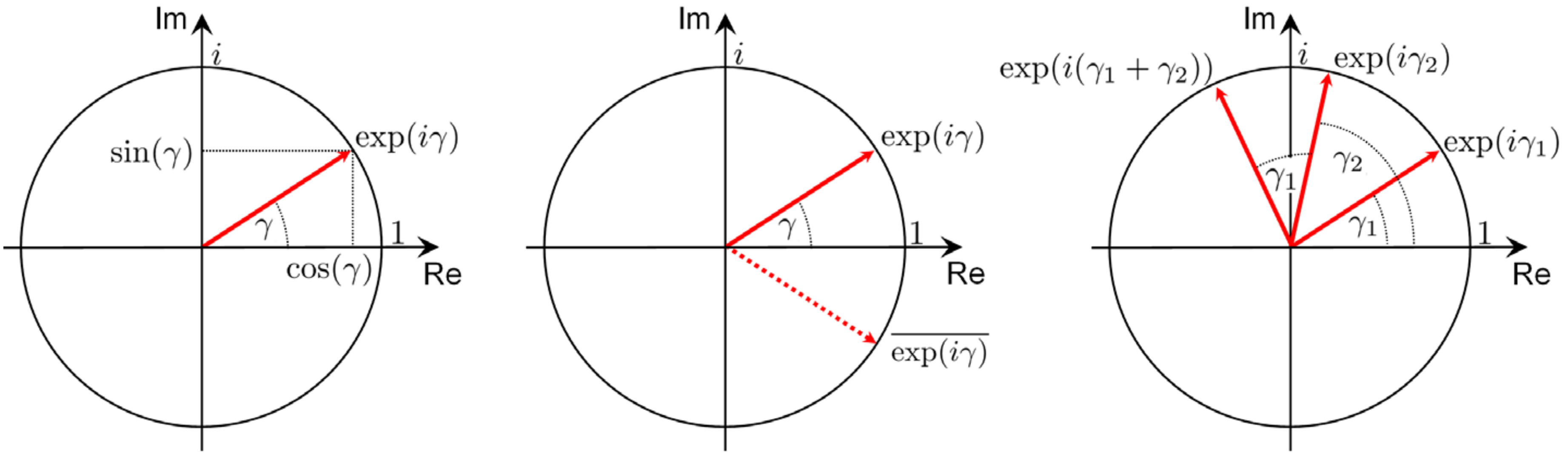

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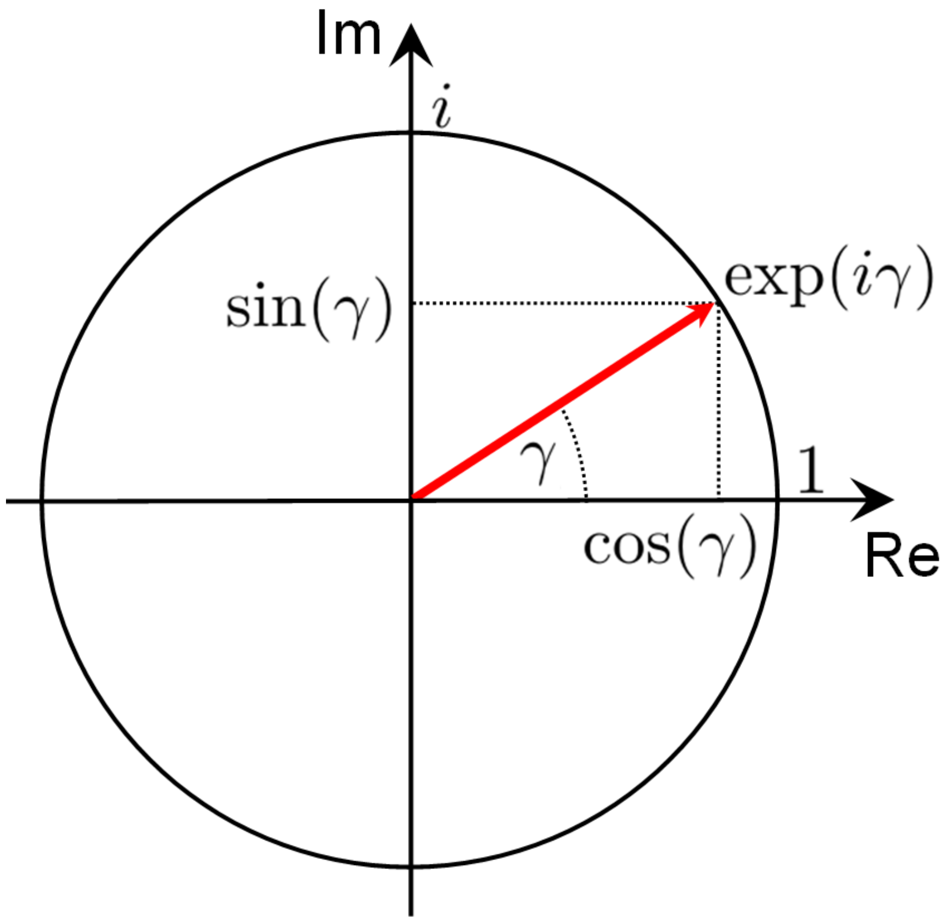

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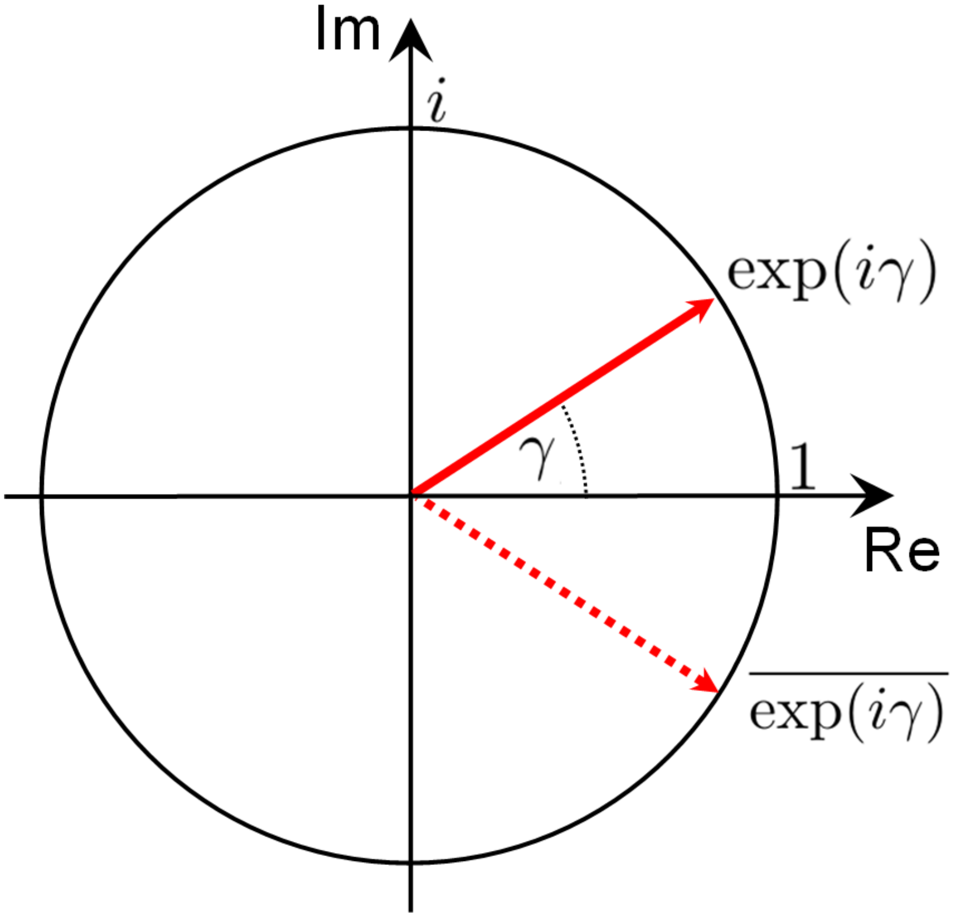

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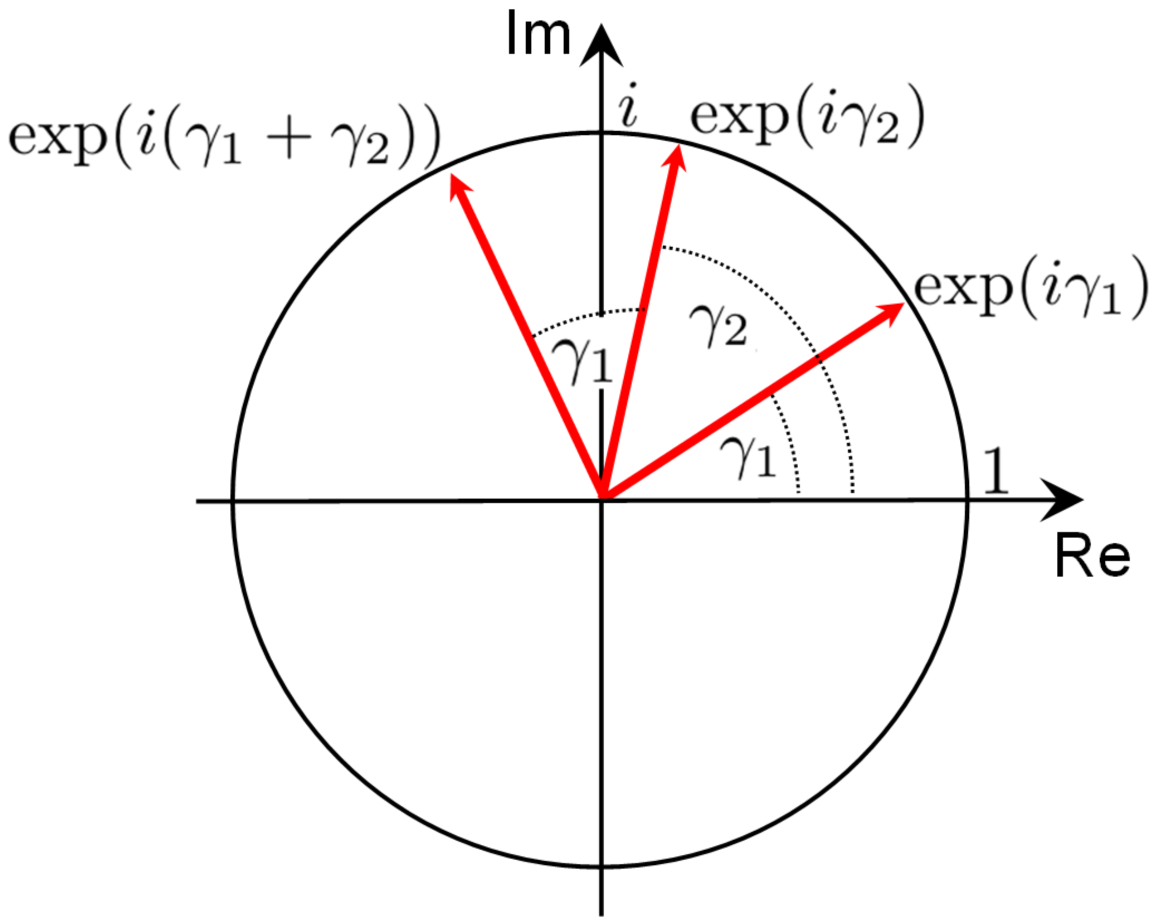

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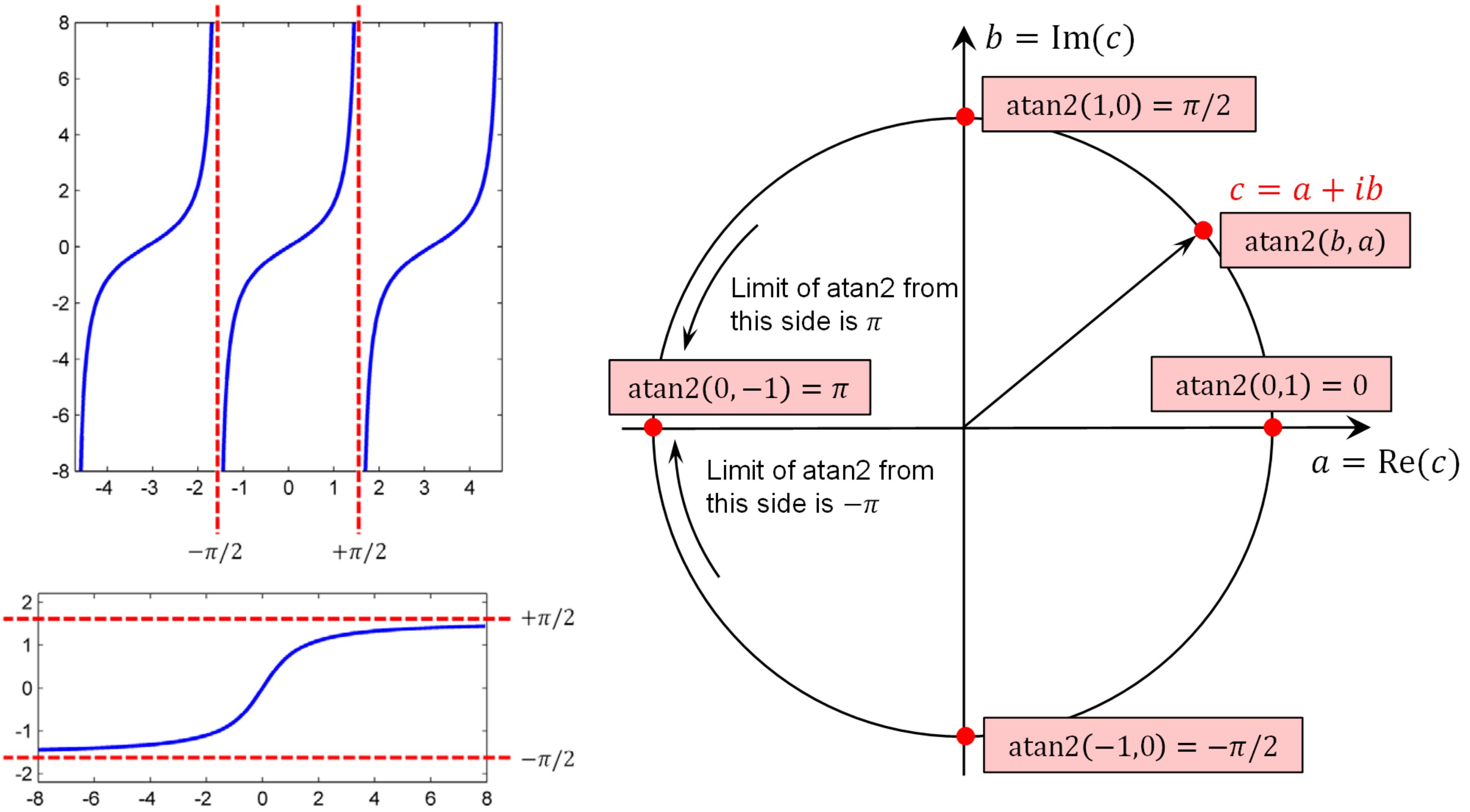
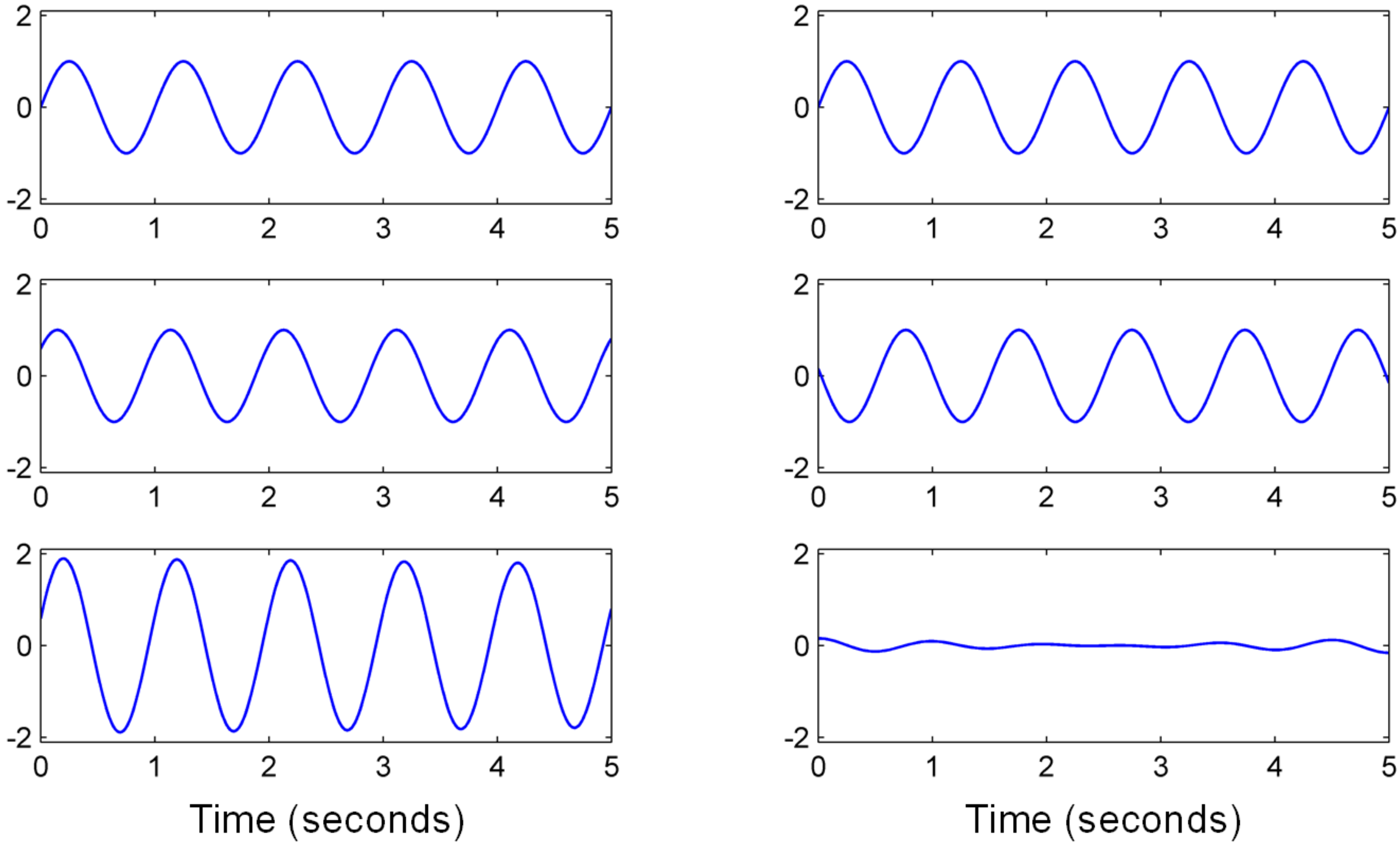


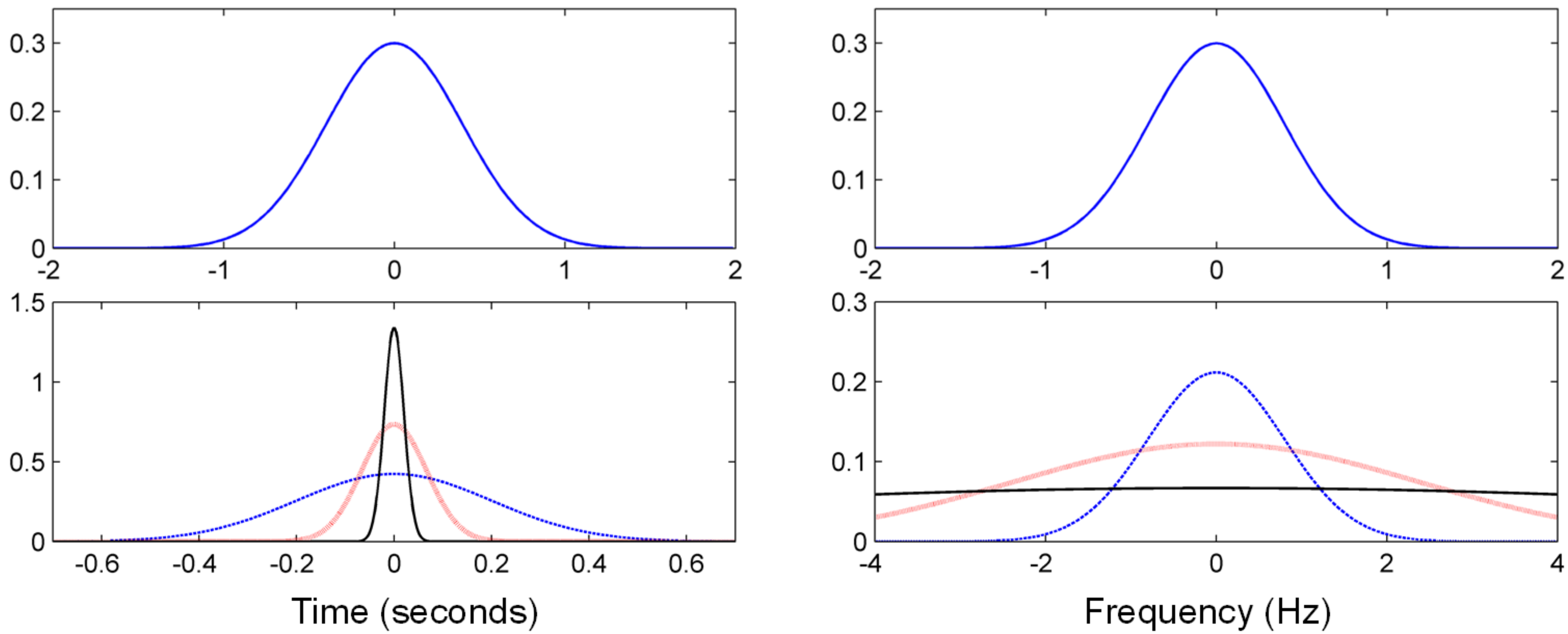
2.3 Fourier Transform

Fig. 2.19



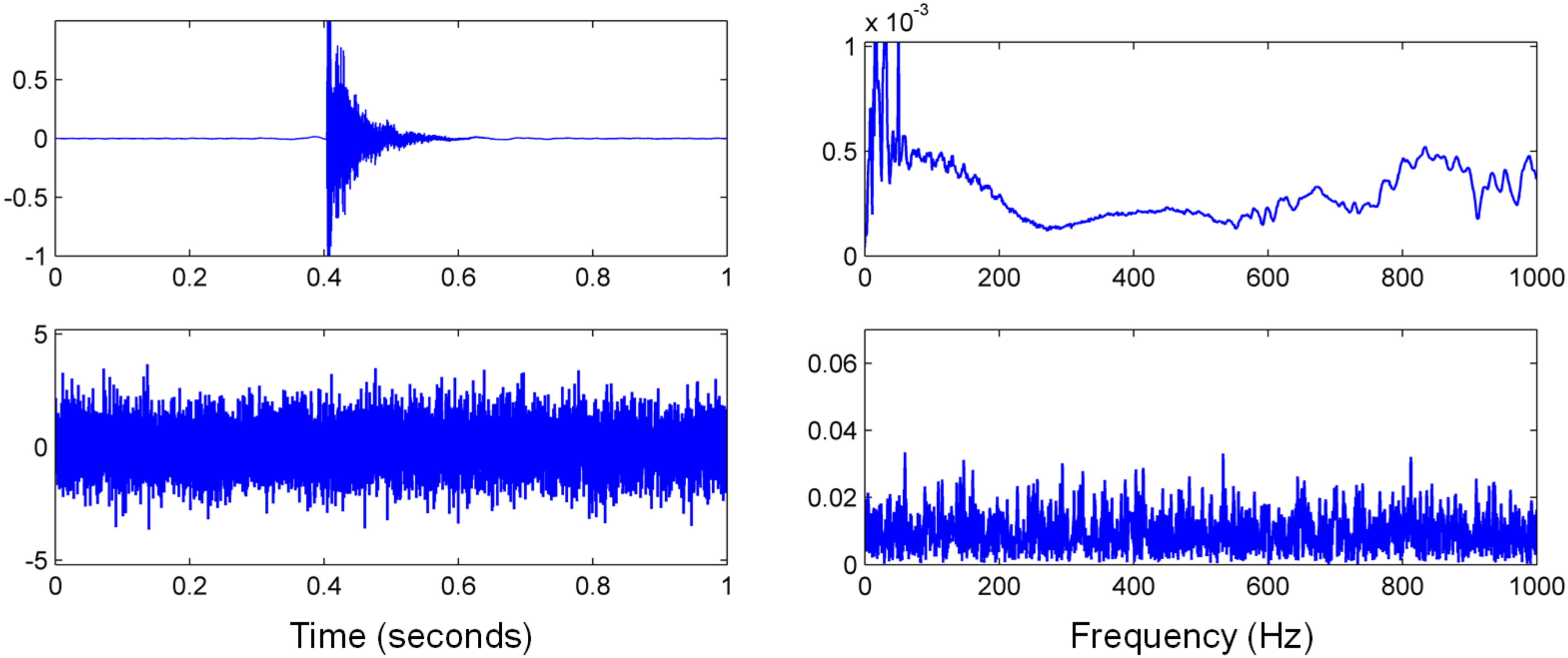
2.3 Fourier Transform

Fig. 2.20



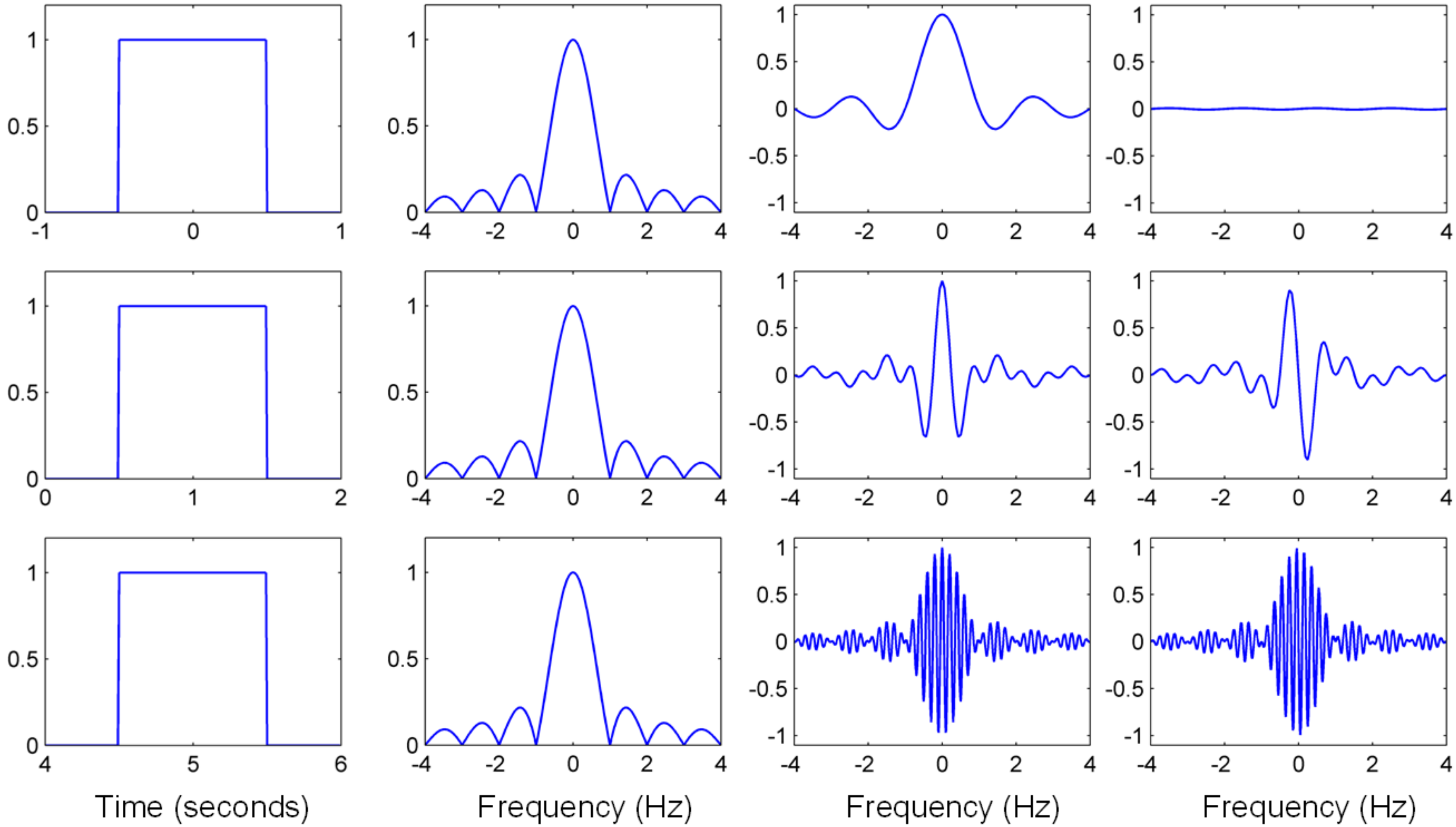
2.3 Fourier Transform

Fig. 2.21



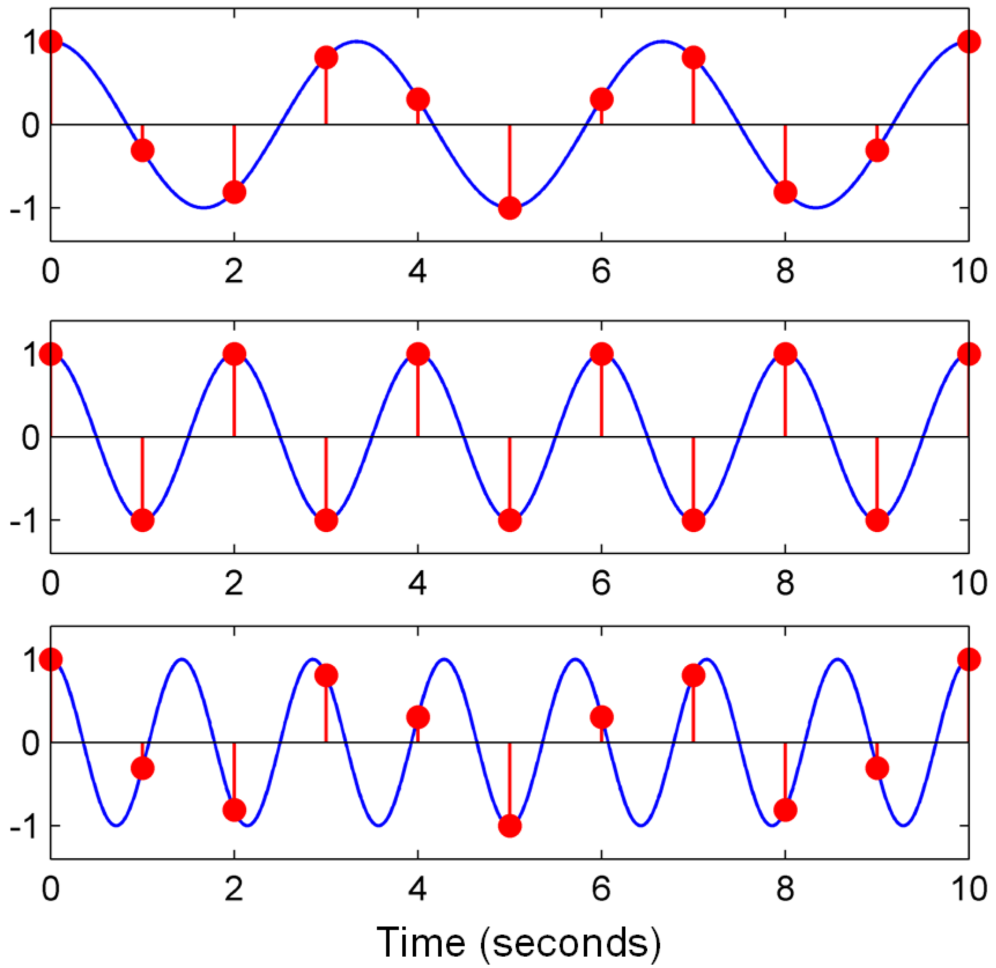
2.3 Fourier Transform

Fig. 2.22



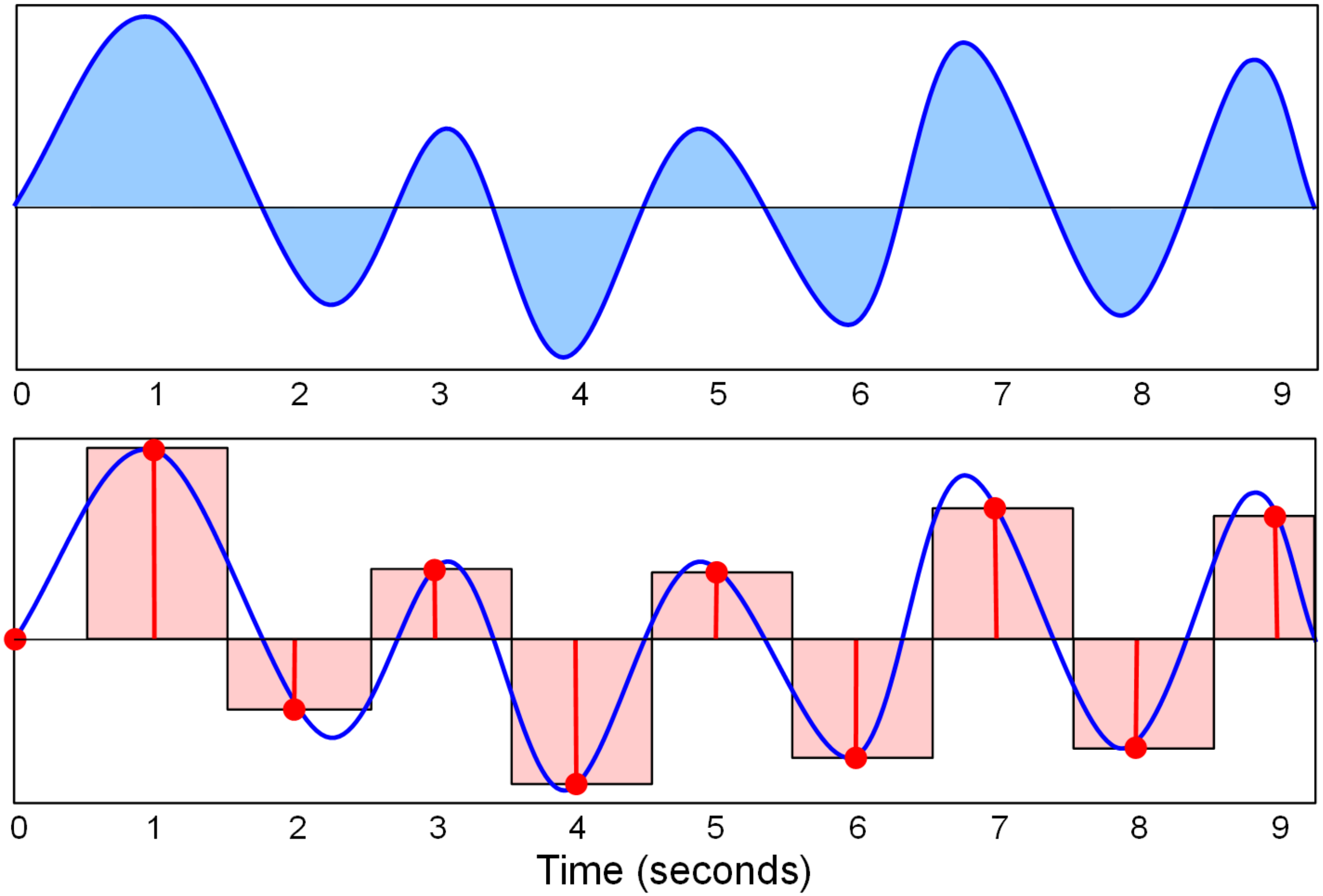
2.3 Fourier Transform

Fig. 2.23



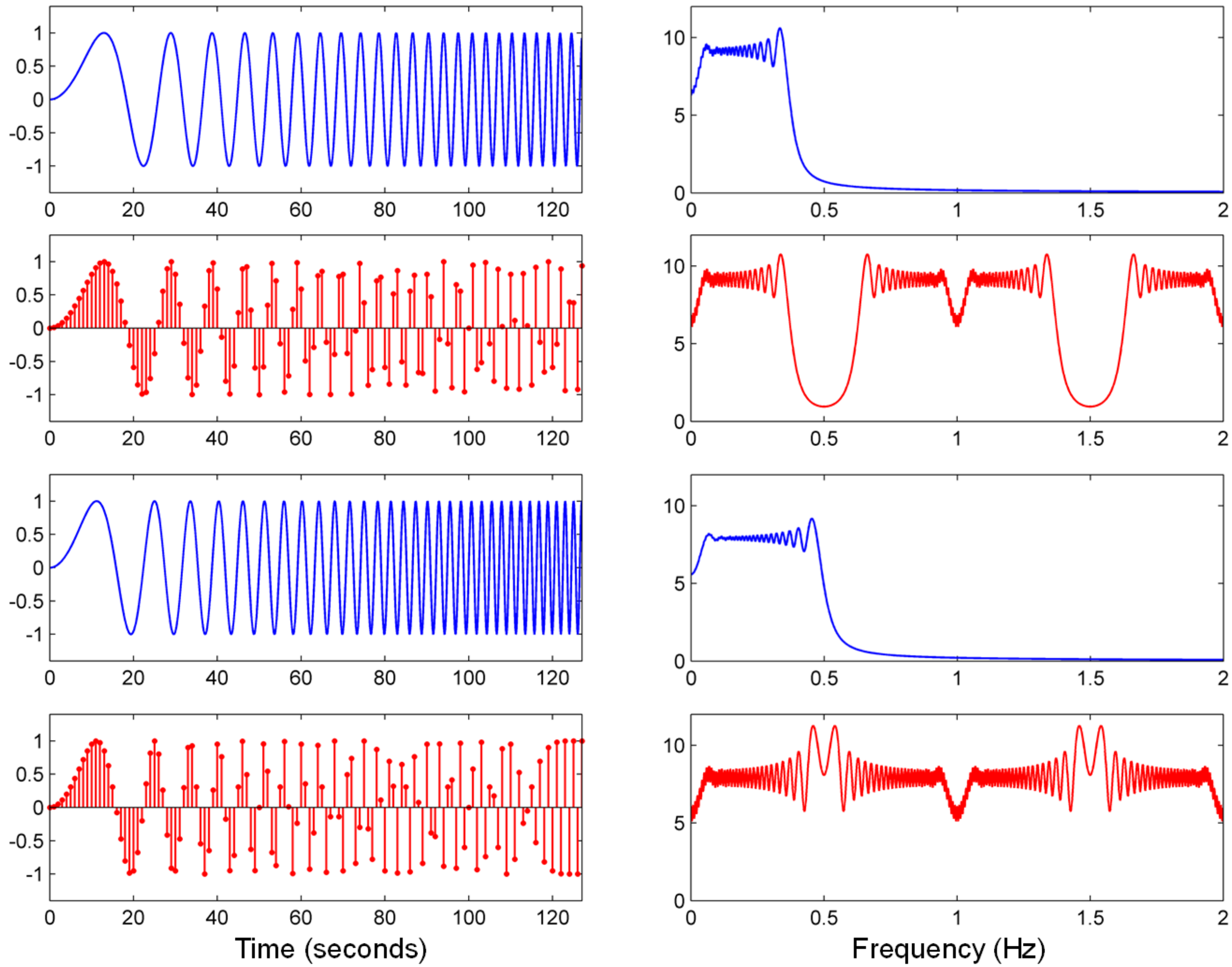
2.3 Fourier Transform

Fig. 2.24



2.3 Fourier Transform

Fig. 2.25



2.3 Fourier Transform

Fig. 2.25

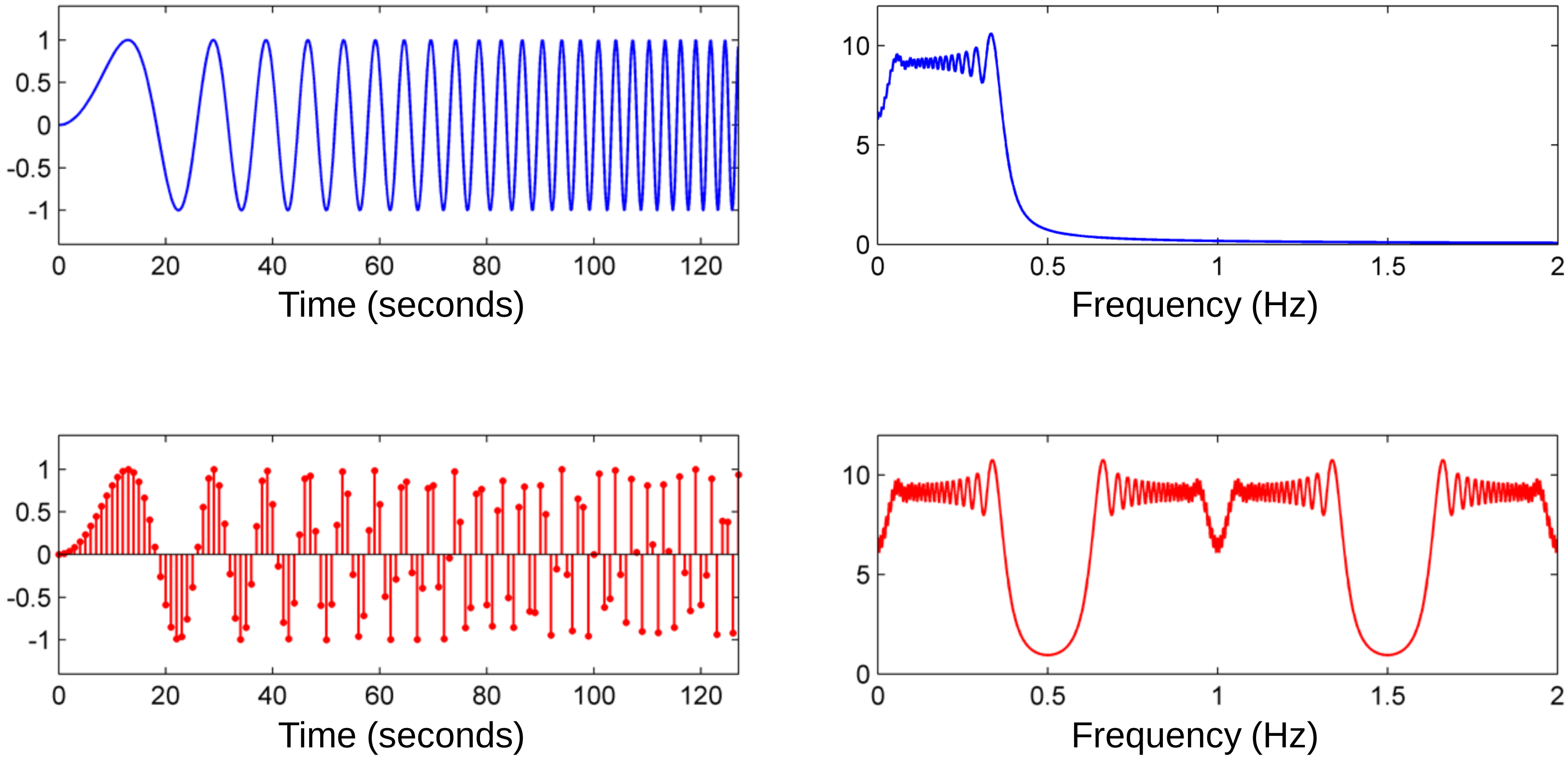


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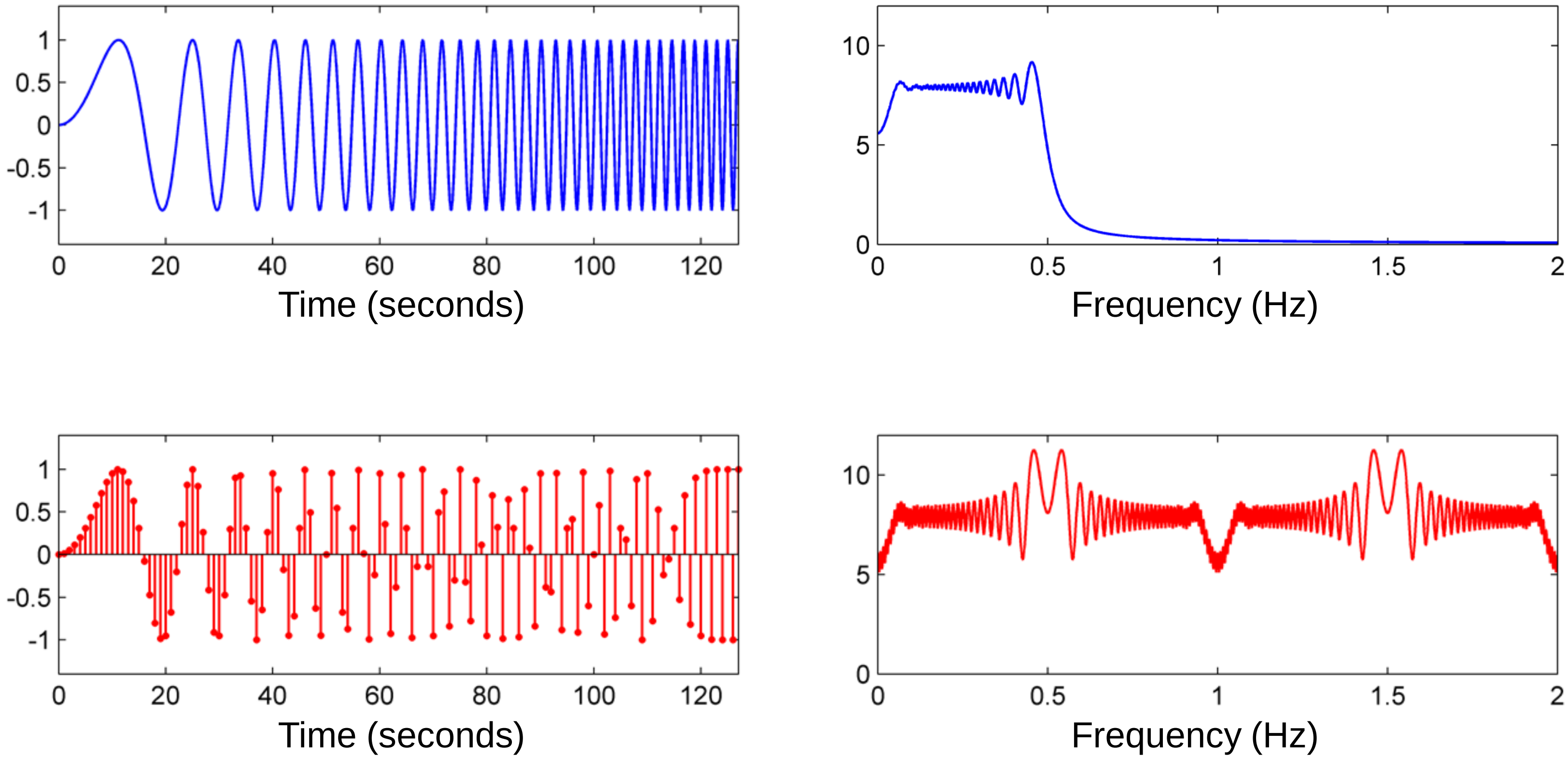
Fig. 2.25



2.4 Discrete Fourier Transform (DFT)

Table 2.1

Algorithm: FFT

Input: The length $N = 2^L$ with N being a power of two
The vector $(x(0),\ldots,x(N-1))^\top \in \mathbb{C}^N$

Output: The vector $(X(0),\ldots,X(N-1))^\top = \mathrm{DFT}_N \cdot (x(0),\ldots,x(N-1))^\top$

Procedure: Let $(X(0),\ldots,X(N-1)) = \mathrm{FFT}(N,x(0),\ldots,x(N-1))$ denote the general form of the FFT algorithm.
If $N = 1$ then

$$X(0) = x(0).$$

Otherwise compute recursively:

$$(A(0),\ldots,A(N/2-1)) = \mathrm{FFT}(N/2,x(0),x(2),x(4)\ldots,x(N-2)),$$

$$(B(0),\ldots,B(N/2-1)) = \mathrm{FFT}(N/2,x(1),x(3),x(5),\ldots,x(N-1)),$$

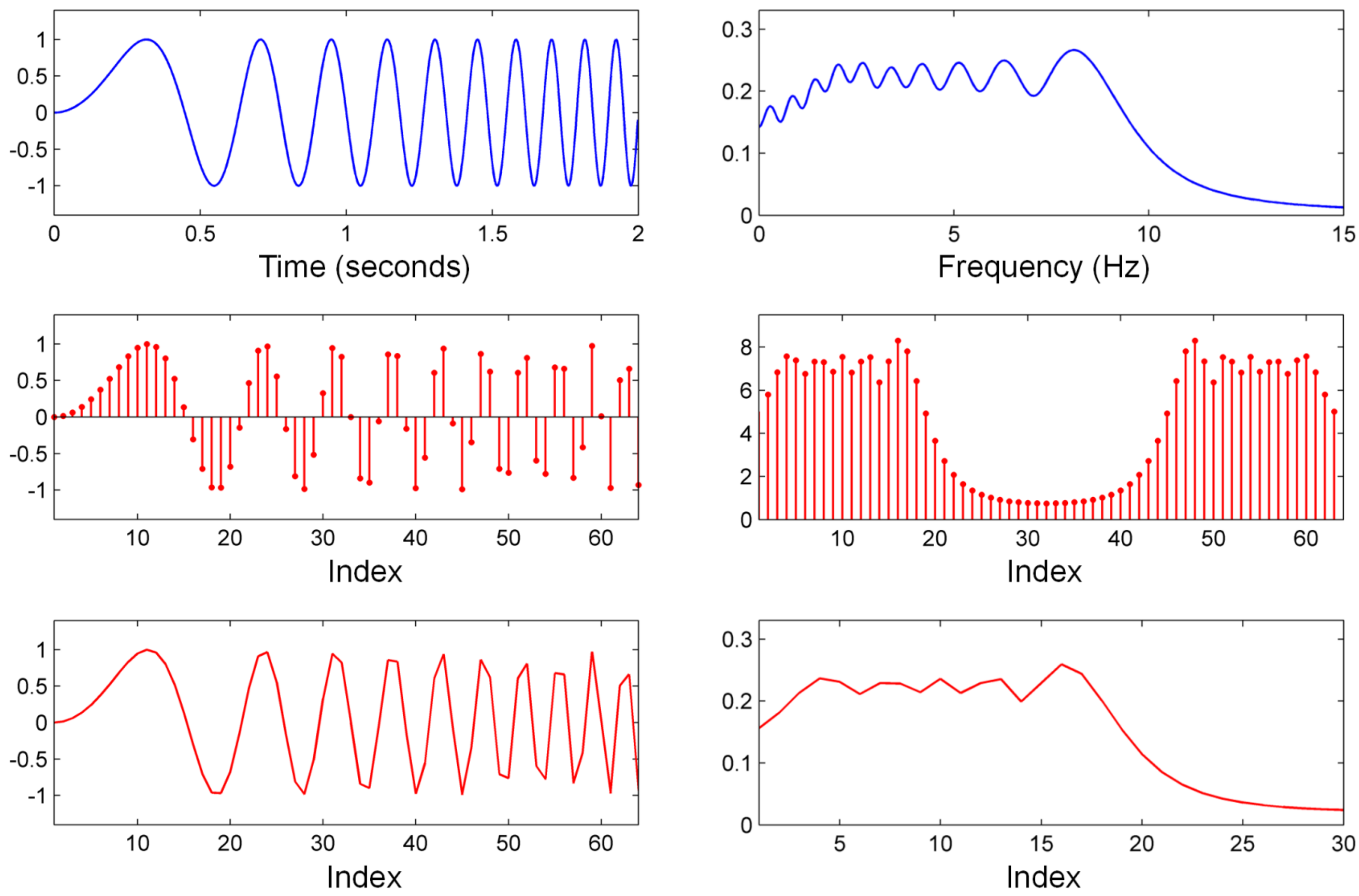
$$C(k) = \omega_N^k \cdot B(k) \text{ for } k \in [0:N/2-1],$$

$$X(k) = A(k)+C(k) \text{ for } k \in [0:N/2-1],$$

$$X(N/2+k) = A(k)-C(k) \text{ for } k \in [0:N/2-1].$$

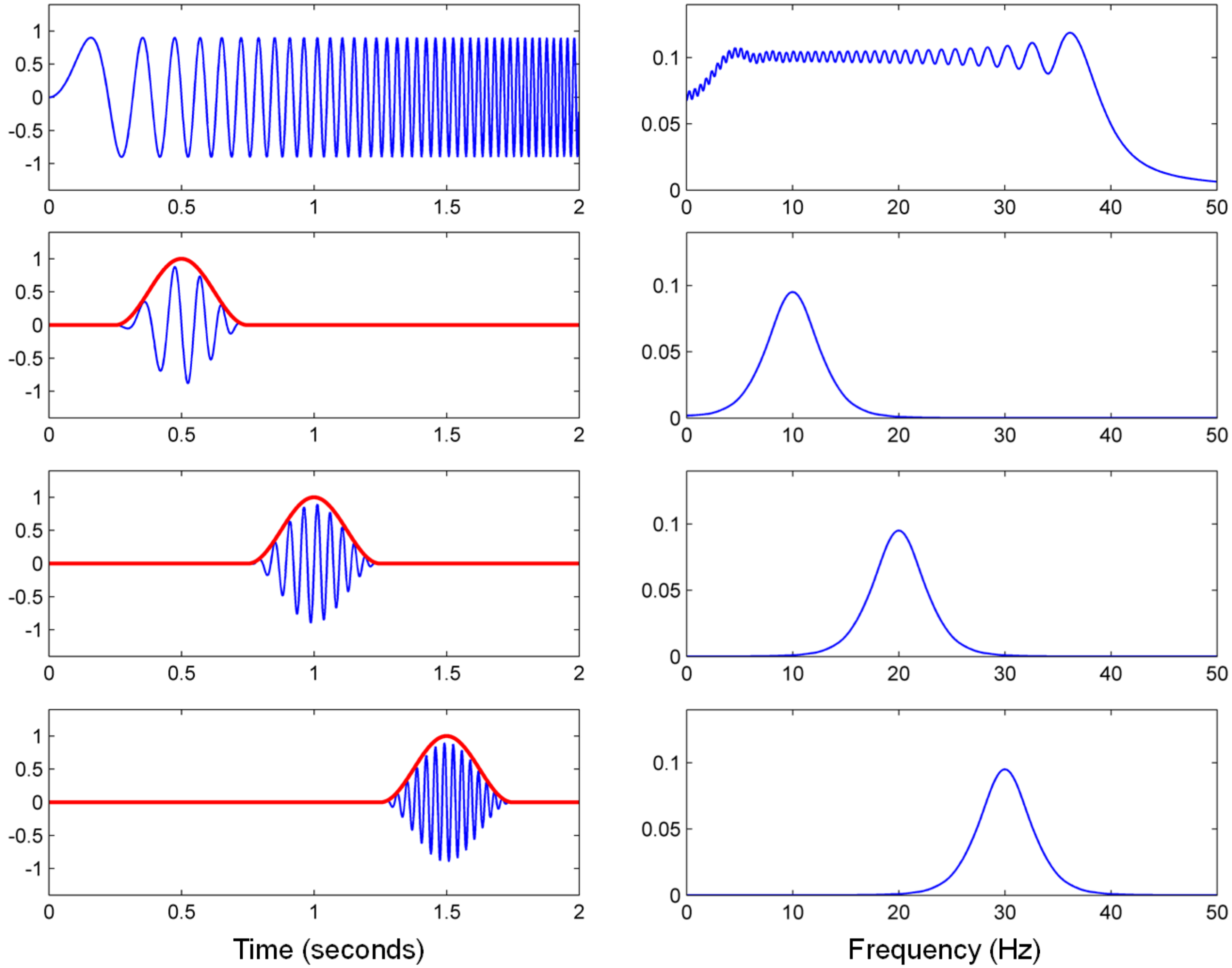
2.4 Discrete Fourier Transform (DFT)

Fig. 2.26



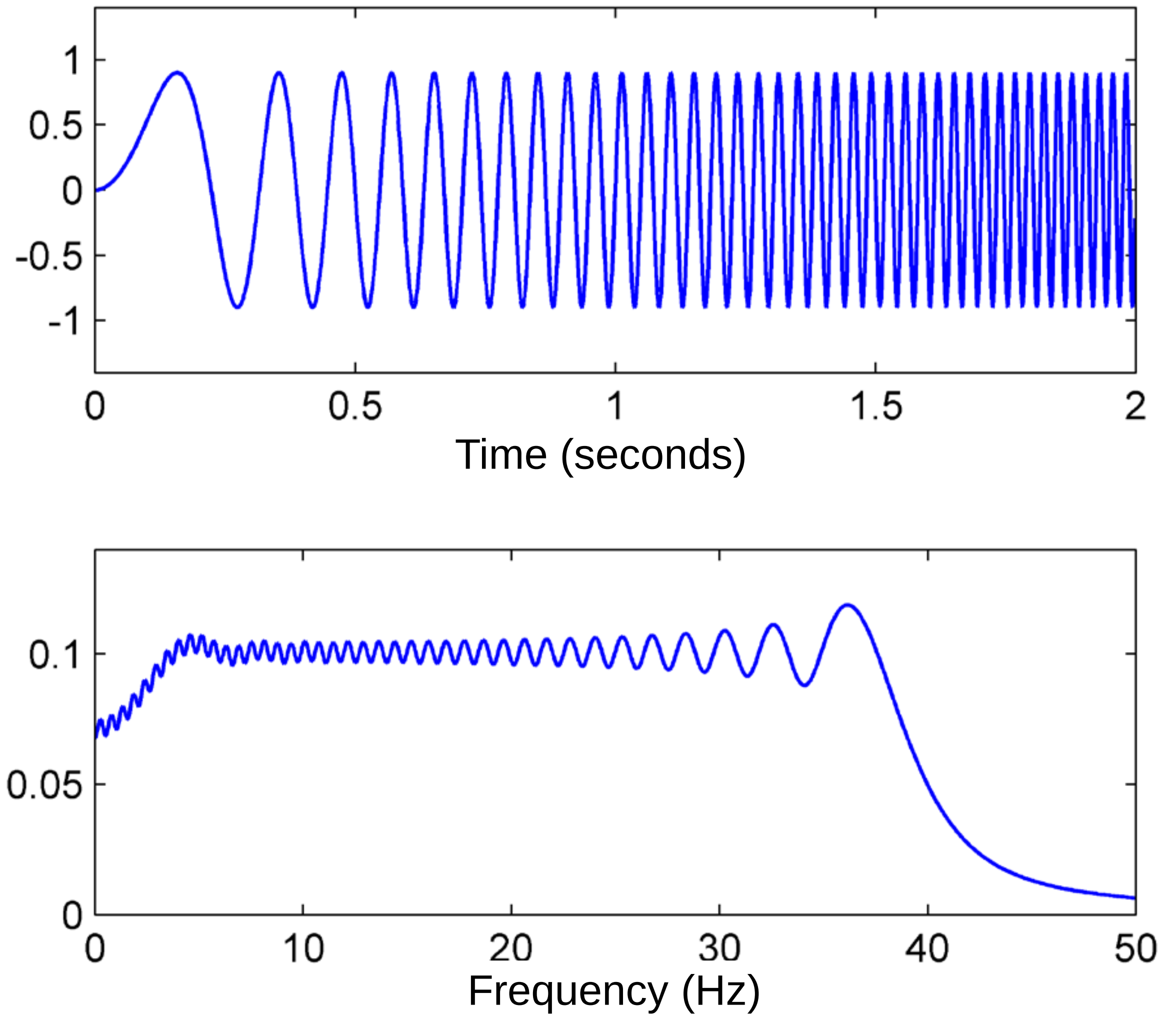
2.5 Short-Time Fourier Transform (STFT)

Fig. 2.27



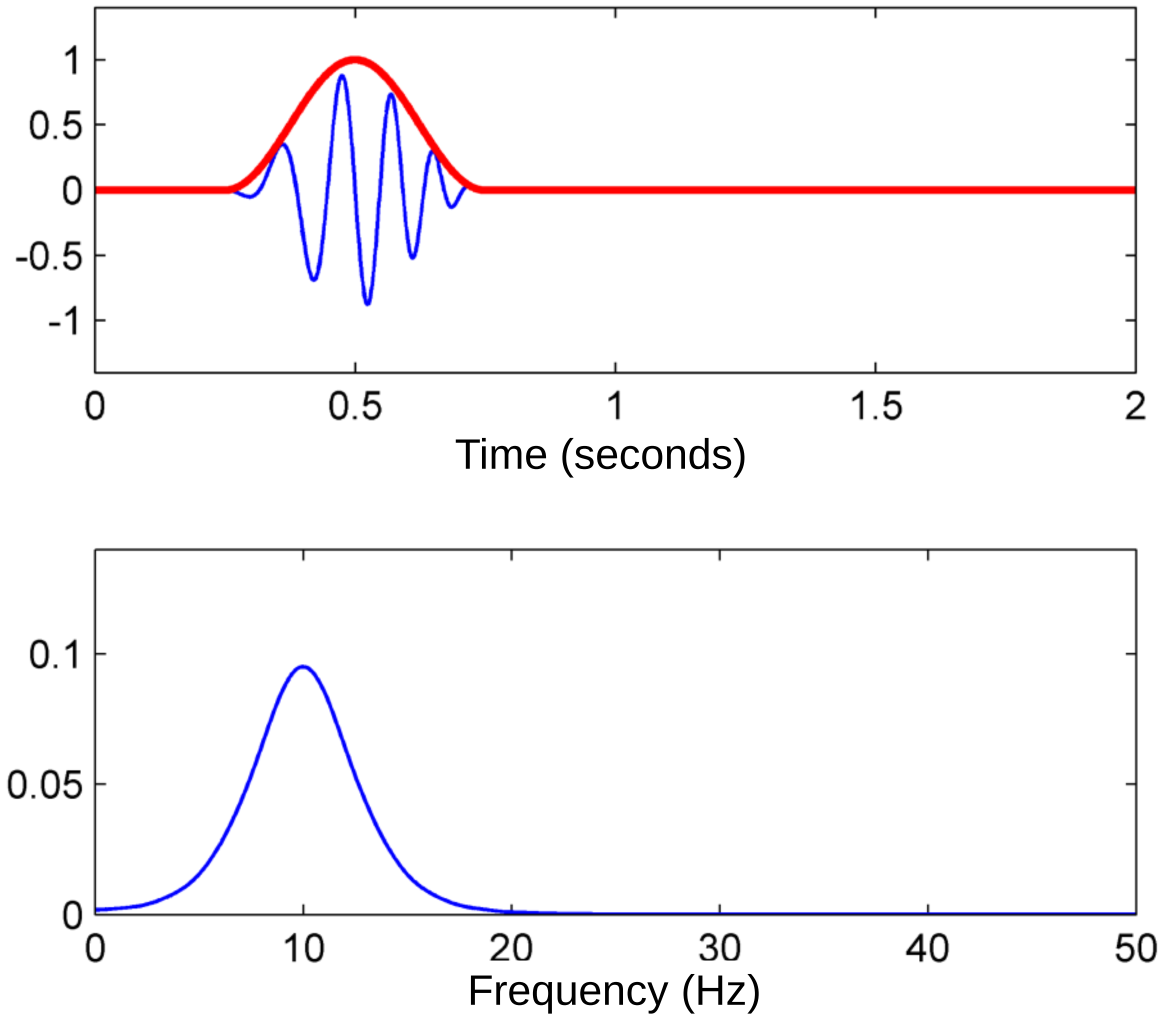
2.5 Short-Time Fourier Transform (STFT)

Fig. 2.27



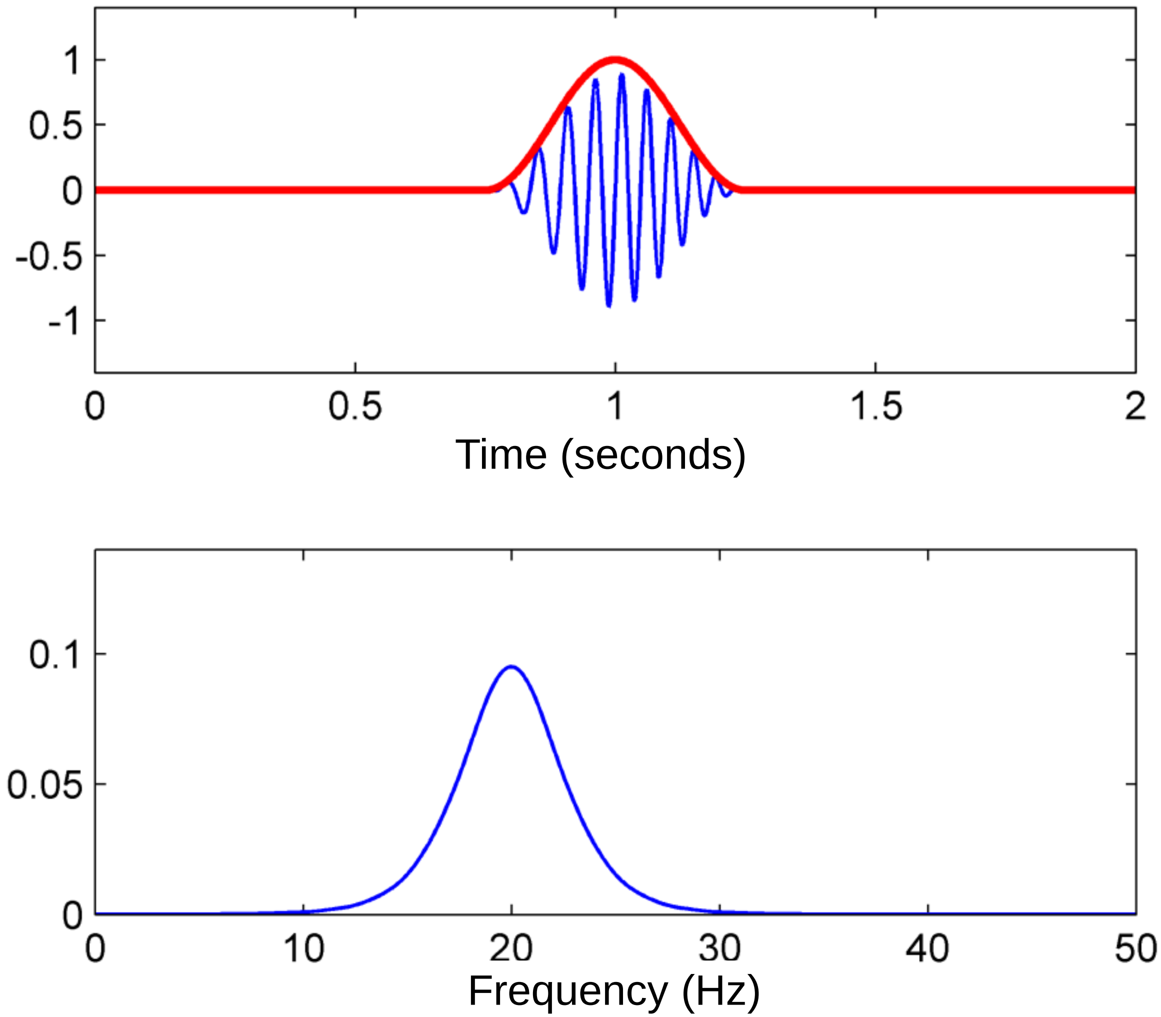
2.5 Short-Time Fourier Transform (STFT)

Fig. 2.27



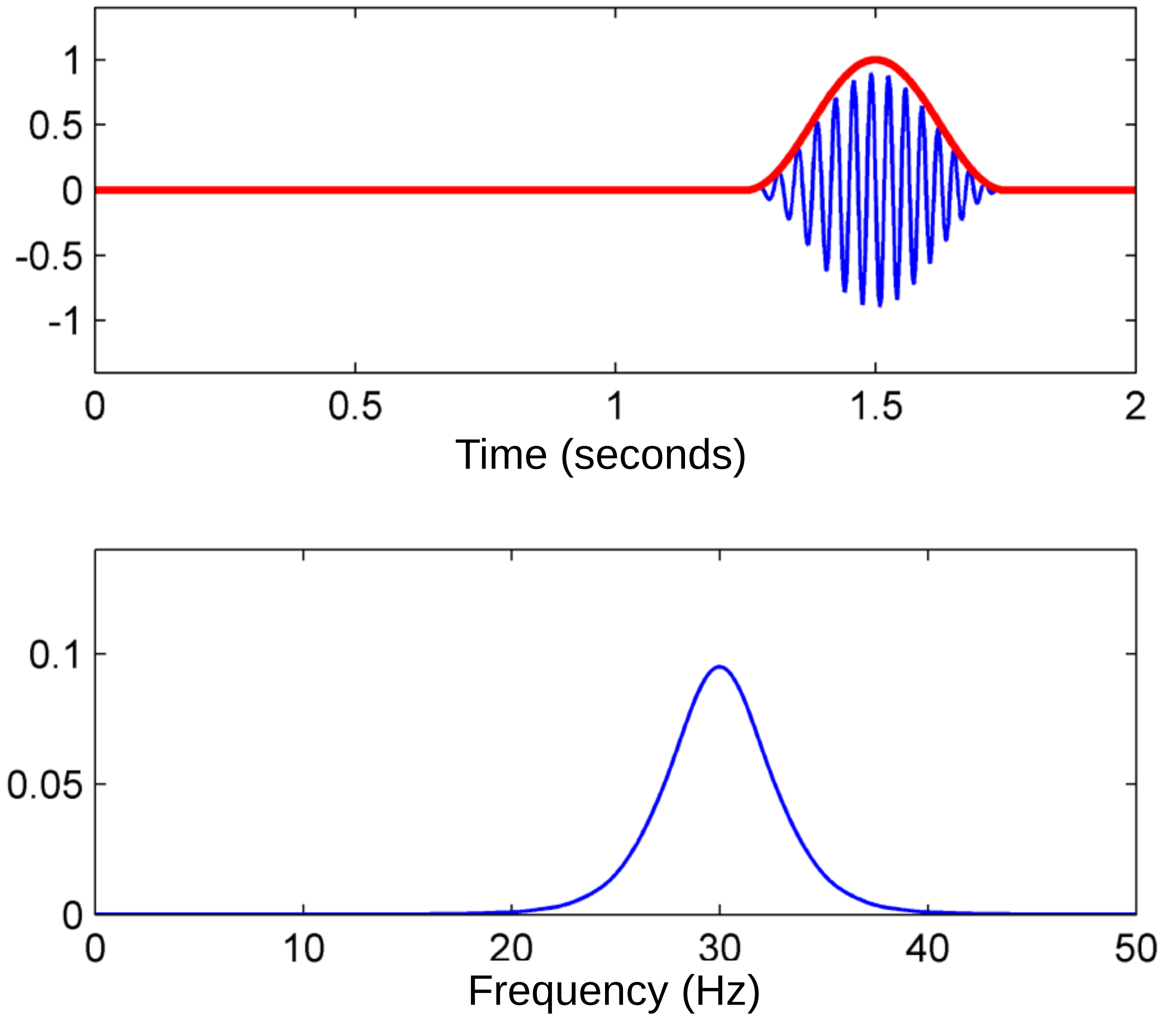
2.5 Short-Time Fourier Transform (STFT)

Fig. 2.27



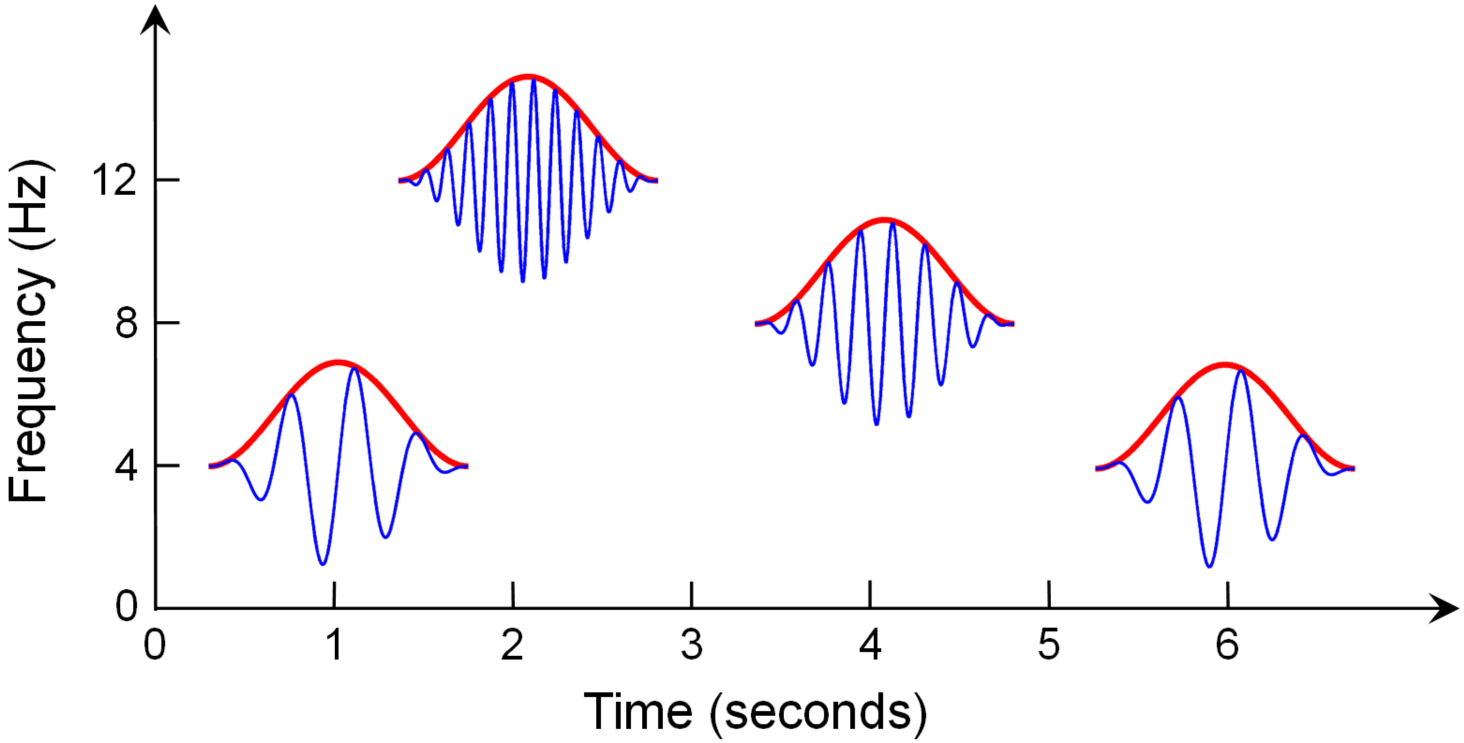
2.5 Short-Time Fourier Transform (STFT)

Fig. 2.27



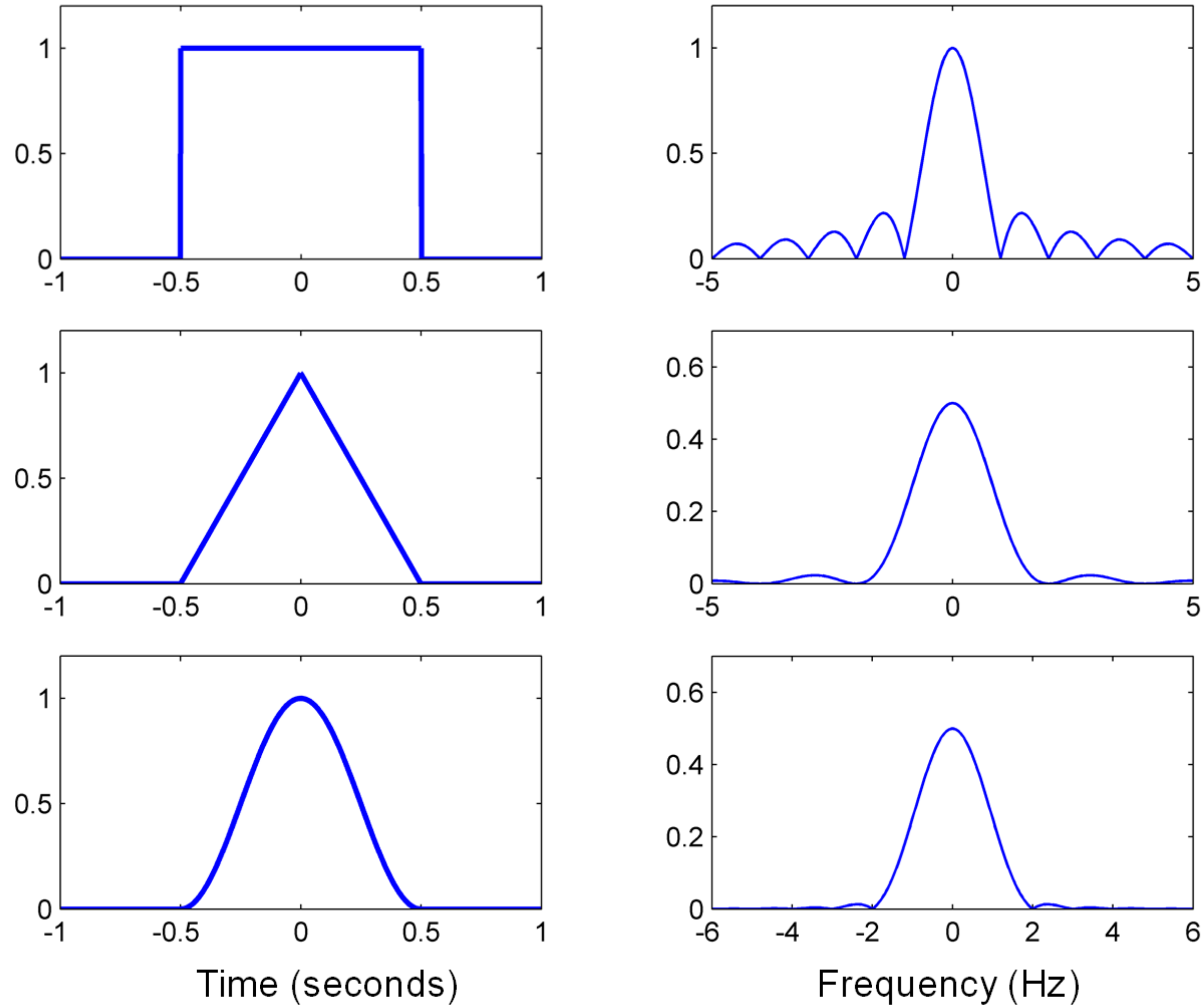
2.5 Short-Time Fourier Transform (STFT)

Fig. 2.28



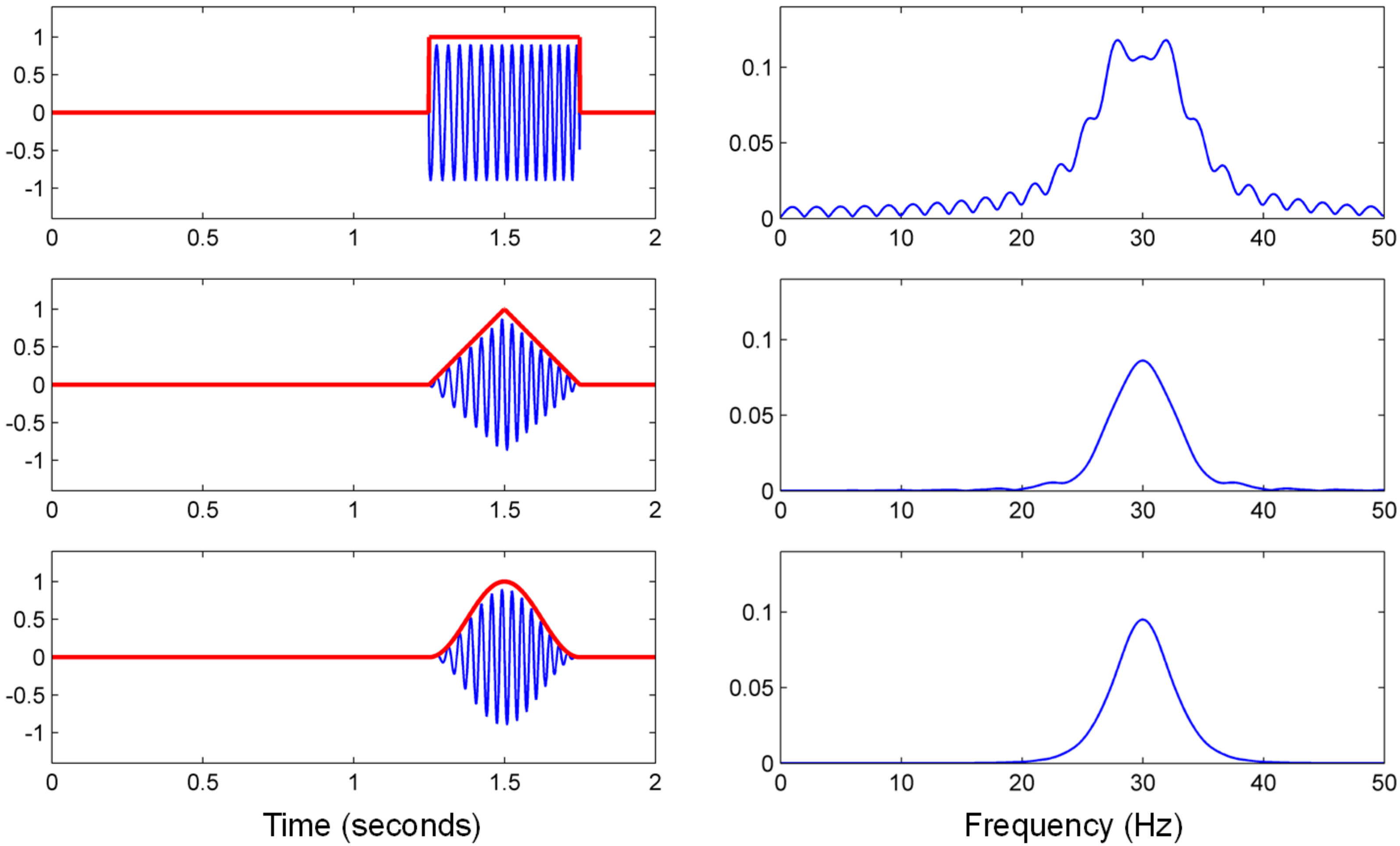
2.5 Short-Time Fourier Transform (STFT)

Fig. 2.29



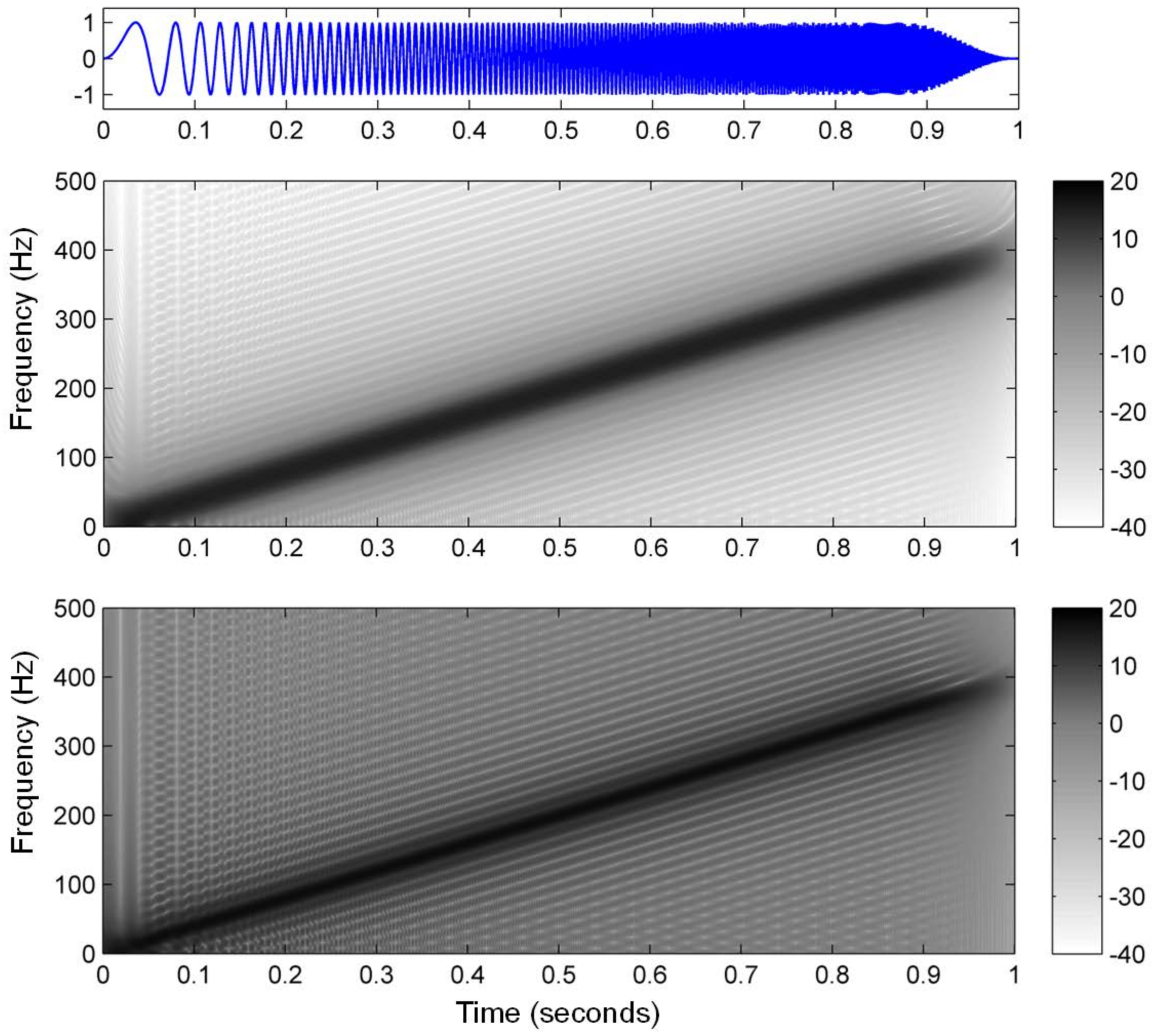
2.5 Short-Time Fourier Transform (STFT)

Fig. 2.30



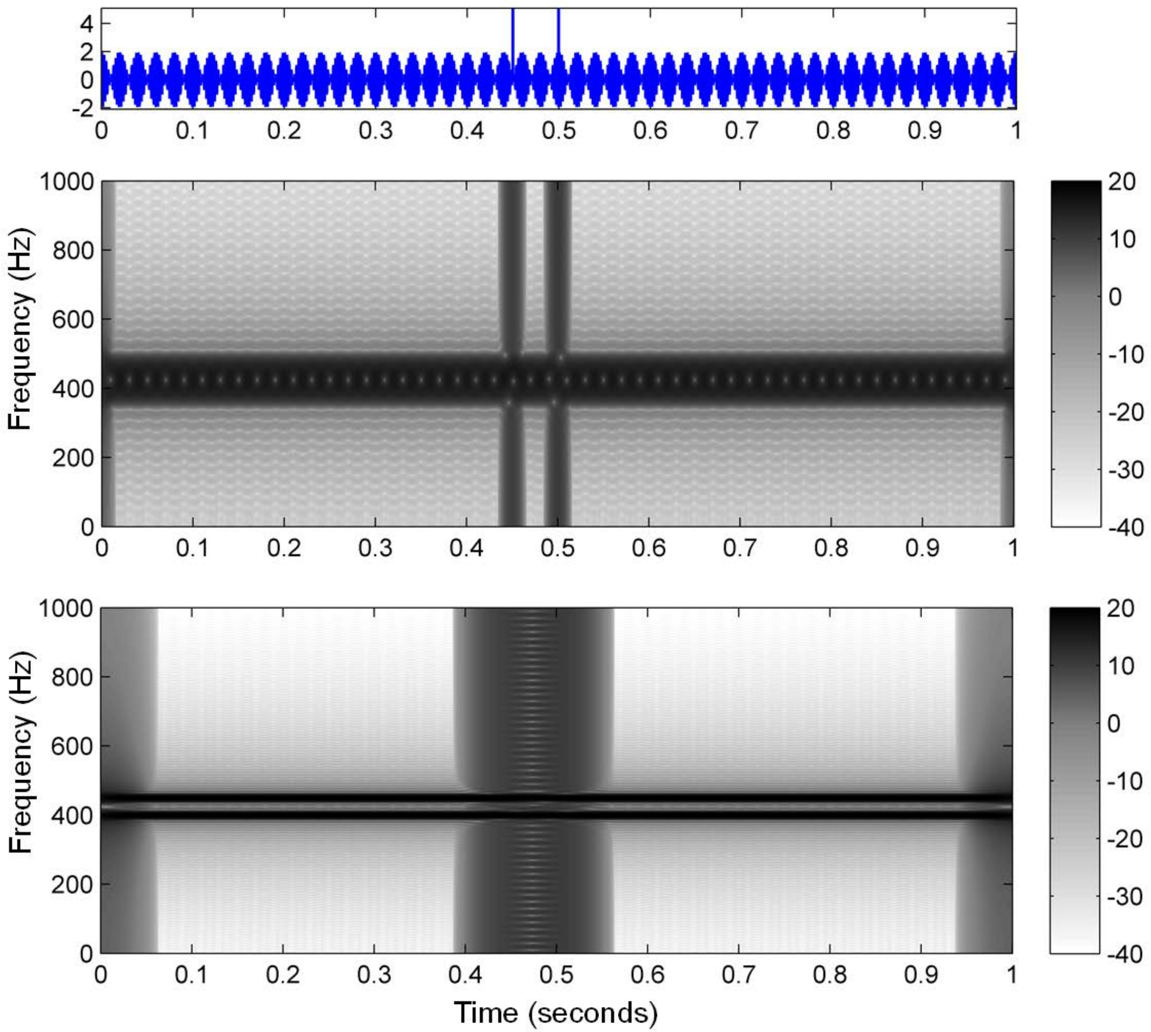
2.5 Short-Time Fourier Transform (STFT)

Fig. 2.31



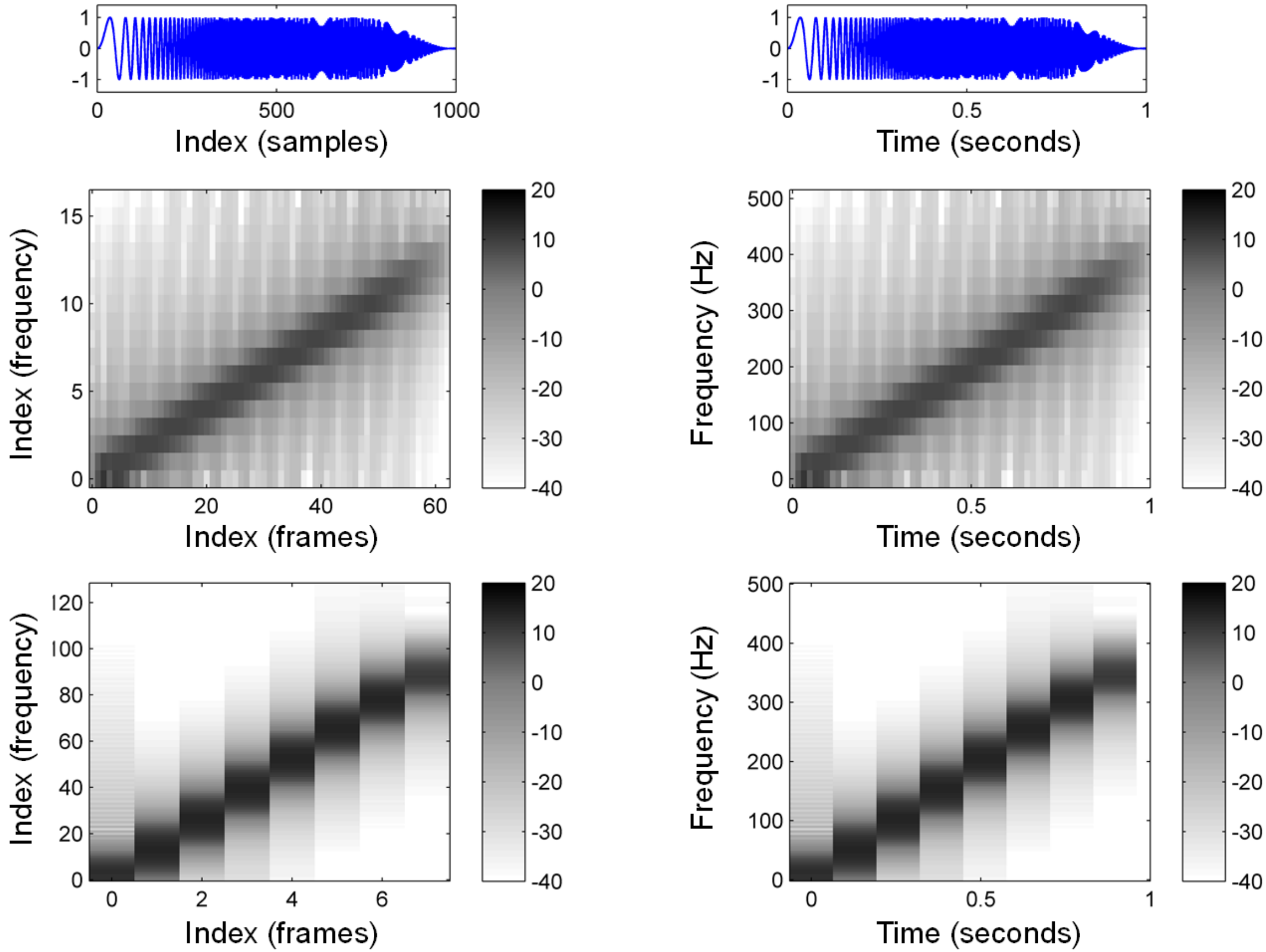
2.5 Short-Time Fourier Transform (STFT)

Fig. 2.32



2.5 Short-Time Fourier Transform (STFT)

Fig. 2.33



2.6 Further Notes

Table 2.2

Signal space	$L^2(\mathbb{R})$	$L^2([0,1))$	$\ell^2(\mathbb{Z})$
Inner product	$\langle f \vert g\rangle = \int_{t\in\mathbb{R}} f(t)\overline{g(t)}dt$	$\langle f \vert g\rangle = \int_{t\in[0,1)} f(t)\overline{g(t)}dt$	$\langle x \vert y\rangle = \sum_{n\in\mathbb{Z}} x(n)\overline{y(n)}$
Norm	$\\|f\\|_2 = \sqrt{\langle f \vert f\rangle}$	$\\|f\\|_2 = \sqrt{\langle f \vert f\rangle}$	$\\|x\\|_2 = \sqrt{\langle x \vert x\rangle}$
Definition	$L^2(\mathbb{R}) :=$ $\{f:\mathbb{R}\to\mathbb{C} \mid \\|f\\|_2 < \infty\}$	$L^2([0,1)) :=$ $\{f:[0,1)\to\mathbb{C} \mid \\|f\\|_2 < \infty\}$	$\ell^2(\mathbb{Z}) :=$ $\{f:\mathbb{Z}\to\mathbb{C} \mid \\|x\\|_2 < \infty\}$
Elementary frequency function	$\mathbb{R}\to\mathbb{C}$ $t \mapsto \exp(2\pi i\omega t)$	$[0,1)\to\mathbb{C}$ $t \mapsto \exp(2\pi ikt)$	$\mathbb{Z}\to\mathbb{C}$ $n \mapsto \exp(2\pi i\omega n)$
Frequency parameter	$\omega\in\mathbb{R}$	$k\in\mathbb{Z}$	$\omega\in[0,1)$
Fourier representation	$f(t) =$ $\int_{\omega\in\mathbb{R}} c_\omega \exp(2\pi i\omega t)d\omega$	$f(t) =$ $\sum_{k\in\mathbb{Z}} c_k \exp(2\pi ikt)$	$x(n) =$ $\int_{\omega\in[0,1)} c_\omega \exp(2\pi i\omega n)d\omega$
Fourier transform	$\hat{f}:\mathbb{R}\to\mathbb{C}$ $\hat{f}(\omega) = c_\omega =$ $\int_{t\in\mathbb{R}} f(t)\exp(-2\pi i\omega t)dt$	$\hat{f}:\mathbb{Z}\to\mathbb{C}$ $\hat{f}(k) = c_k =$ $\int_{t\in[0,1)} f(t)\exp(-2\pi ikt)dt$	$\hat{x}:[0,1)\to\mathbb{C}$ $\hat{x}(\omega) = c_\omega =$ $\sum_{n\in\mathbb{Z}} x(n)\exp(-2\pi i\omega n)$