Relation between CT Fourier transform and DFT

$$f : \mathbb{R} \to \mathbb{R}, \quad f \in L^2(\mathbb{R}) \qquad \text{CT-signal}$$

$$\hat{f}(\omega) = \int_{t \in \mathbb{R}} f(t) \cdot \exp(-2\pi i \omega t)\, dt$$

Sampling: $x(n) = f(nT), \quad n \in \mathbb{Z}, \quad F_s = \frac{1}{T}$ sampling rate

Windowing: $\vec{x} = (x(0), x(1), \ldots, x(N-1))^T \in \mathbb{R}^N, \quad N =$ window size

DFT: $$X(k) = \sum_{n=0}^{N-1} x(n) \exp(-2\pi i\, n \cdot k/N)$$

$x(n) = f(nT),\ \omega = \frac{k}{N}$

$$\overset{(1)}{\approx} \sum_{n \in \mathbb{Z}} f(nT) \exp(-2\pi i\, n \cdot \omega)$$

(assuming $x(n) \approx 0$ outside window)

$$\overset{(2)}{\approx} \int_{t \in \mathbb{R}} f(tT) \exp(-2\pi i\, t\, \omega)$$

substitution $\varphi(t) = tT$, $\varphi'(t) = T$

$$= \int_{t \in \mathbb{R}} \frac{1}{T} f(t) \exp\left(-2\pi i\, t \cdot \frac{\omega}{T}\right)$$

$\int_{t \in \Omega} g(\varphi(t)) \cdot \varphi'(t)\, dt = \int_{t \in \varphi(\Omega)} g(t)\, dt$

$$= \frac{1}{T} \hat{f}\left(\frac{\omega}{T}\right)$$

$$= F_s \cdot \hat{f}\left(\frac{F_s}{N} \cdot k\right)$$

Interpretation of k^{th} Fourier coefficient:

$$F_{coef}(k) = \frac{k \cdot F_s}{N} \quad (Hz)$$

Frequency resolution of DFT: $\frac{F_s}{N}$

(1) Good if $x(n) \approx 0$ for $n \in \mathbb{Z} \setminus [0 : N-1]$

(2) Riemannian approximation; good if f contains no high frequencies and k is small

Note: f real-valued $\Rightarrow X(k) = \overline{X(N-k)}$

Only $X(0), X(1), \ldots, X(\lfloor \frac{N}{2} \rfloor)$ are meaningful, $\quad k = \frac{N}{2}$ Nyquist rate

Example: $F_s = 44100,\ N = 1024,\ F_{coef}(k) = 43 \cdot k$ Hz

$F_s = 1000,\ N = 100,\ F_{coef}(k) = 10 \cdot k$