Fouriertransform as "optimal solution"

Fourier representation: $f(t) = \int_{\omega \in \mathbb{R}_{\geq 0}} d\omega \quad \sqrt{2}\cos(2\pi(\omega t - \varphi_\omega)) \, d\omega$

Fourier transform : $c_\omega := \hat{f}(\omega) := \int_{t \in \mathbb{R}} f(t) \exp(-2\pi i \omega t) dt = |c_\omega| \exp(i \gamma_\omega)$

Claim: $d_\omega := \max_{\varphi \in [0,1]} \left(\int_t f(t) \sqrt{2} \cos(2\pi(\omega t - \varphi)) \right) dt = \sqrt{2}|c_\omega|$

$\varphi_\omega := \operatorname{argmax}_{\varphi \in [0,1]} \left(\int_t f(t) \sqrt{2} \cos(2\pi \omega t - \varphi) \right) dt = -\frac{1}{2\pi}\gamma_\omega$

Proof $g : \mathbb{R} \to \mathbb{R}$

$$g(\varphi) := \int_{t \in \mathbb{R}} f(t) \sqrt{2} \cos(2\pi\omega t - 2\pi\varphi) \, dt$$
$$= \sqrt{2} \int_{t \in \mathbb{R}} f(t) \left(\cos(2\pi\omega t) \cos(-2\pi\varphi) - \sin(2\pi\omega t) \sin(-2\pi\varphi) \right) dt$$
$$= \sqrt{2} \cos(2\pi \cdot \varphi) \underbrace{\int_{t \in \mathbb{R}} f(t) \cos(2\pi\omega t)}_{Re(c_\omega)} + \sqrt{2} \sin(2\pi\varphi) \underbrace{\int_{t \in \mathbb{R}} f(t) \sin(2\pi\omega t) \, dt}_{-Im(c_\omega)}$$

$$\frac{\partial g}{\partial \varphi}\Big|_\varphi = -\sqrt{2}\, 2\pi \sin(2\pi\varphi) \, Re(c_\omega) - \sqrt{2}\, 2\pi \cos(2\pi\varphi) \, Im(c_\omega)$$

$$\frac{\partial g}{\partial \varphi}\Big|_\varphi = 0 \iff -\sin(2\pi\varphi) \, Re(c_\omega) = \cos(2\pi\varphi) \, Im(c_\omega)$$
$$\iff \frac{\sin(-2\pi\varphi)}{\cos(-2\pi\varphi)} = \frac{Im(c_\omega)}{Re(c_\omega)}$$
$$\iff \tan(-2\pi\varphi) = \frac{Im(c_\omega)}{Re(c_\omega)}$$
$$\iff \varphi = -\frac{1}{2\pi} \arctan\left(\frac{Im(c_\omega)}{Re(c_\omega)} \right)$$
$$\iff \varphi = -\frac{1}{2\pi} \gamma_\omega$$

Hence: Optimal $\varphi_\omega = -\frac{1}{2\pi}\gamma_\omega$

$$g(\varphi_\omega) = \sqrt{2} \cos\left(2\pi\left(-\tfrac{1}{2\pi}\gamma_\omega\right)\right) Re(c_\omega) + \sqrt{2} \sin\left(2\pi\left(-\tfrac{1}{2\pi}\gamma_\omega\right)\right)(-Im(c_\omega))$$
$$= \sqrt{2} \cos(\gamma_\omega) \, Re(c_\omega) + \sqrt{2} \sin(\gamma_\omega) \, Im(c_\omega)$$
$$= \sqrt{2} \frac{Re(c_\omega)}{|c_\omega|} \cdot Re(c_\omega) + \sqrt{2} \frac{Im(\gamma_\omega)}{|c_\omega|} Im(c_\omega)$$
$$= \sqrt{2} \left(Re(c_\omega)^2 + Im(c_\omega)^2 \right) \cdot \frac{1}{|c_\omega|}$$
$$= \sqrt{2} \, |c_\omega|^2 \frac{1}{|c_\omega|} = \sqrt{2} \, |c_\omega|$$